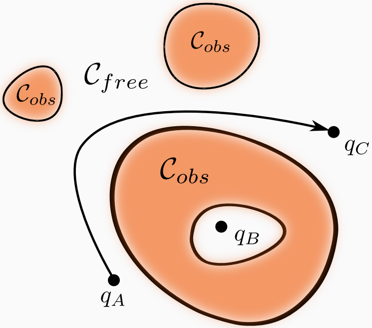
Connectivity in real algebraic sets:
algorithms and applications

11$^{\text{th}}$ March 2024

AROMATH Seminar



Rémi Prébet

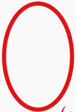
Joint works with M. Safey El Din, É. Schost
N. Islam, A. Poteaux
D.Chablat, D.Salunkhe, P. Wenger

Slides:
rprebet.github.io/#talks

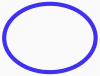
Semi-algebraic sets

Set of **real** solutions of systems of **polynomial equations** and **inequalities**

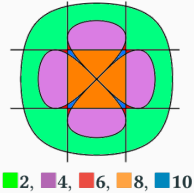
$$\begin{cases} 4y + x^3 - 4x^2 - 2x - 8 = 0 \\ -2 \leq x \leq 0 \end{cases}$$



$$\frac{x^2}{4} + y^2 - 1 = 0$$

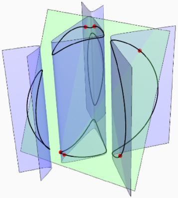


$$(x-1)^2 + \frac{(y-1)^2}{9} - 1 = 0$$



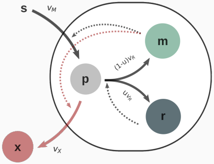
Physics

[Le, Safey El Din; '22]



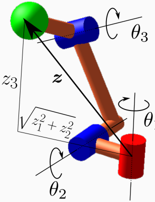
Computational geometry

[Le, Manevich, Plaumann; '21]



Biology

[Yabo, Safey El Din, Caillau, Gouzé; '23]



Robotics

[Chablat, P., Safey El Din, Salunkhe, Wenger; '22]

Computational real algebraic geometry

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Stability [Tarski-Seidenberg]

The family of s.a. sets is stable by projection

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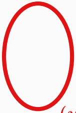
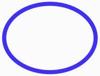


Finiteness

Finite number of connected components

$$\frac{x^2}{4} + y^2 - 1 = 0$$



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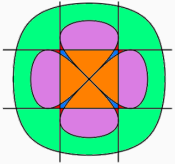

2, 4, 6, 8, 10

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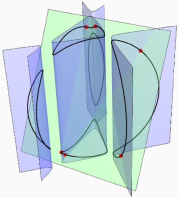

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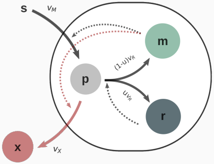

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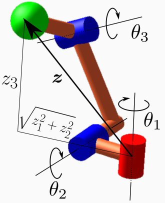

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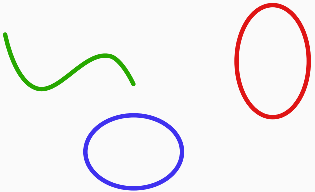

Fundamental problems in computational real algebraic geometry

(P) compute a projection: one block quantifier elimination

(S) compute at least one point in each connected component

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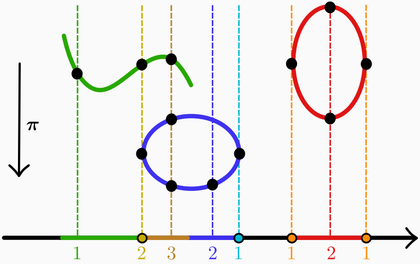

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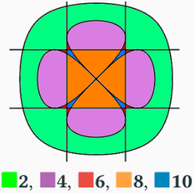

Kuramoto oscillators

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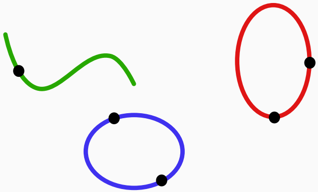

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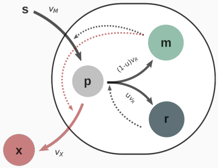

Dynamical systems

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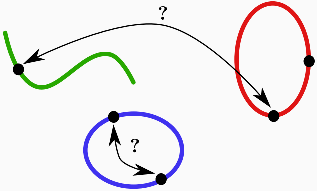

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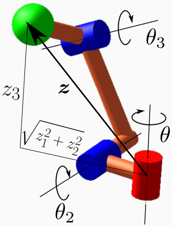

Cuspidality decision

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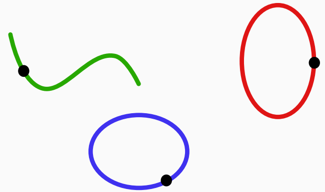

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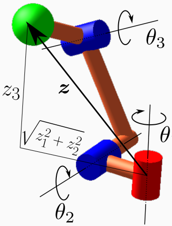

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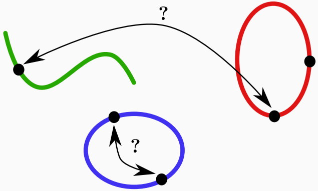

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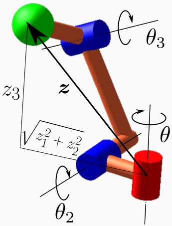

Cuspidality decision

General approach: complete description of the geometry

Input

$S \subset \mathbb{R}^n$ s.a. set defined by s polynomials of deg $\leq D$

$\longrightarrow$

Output

Complete and tractable description of the geometry of S

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Partition of $\mathbb{R}^n$ into semi-algebraic cells
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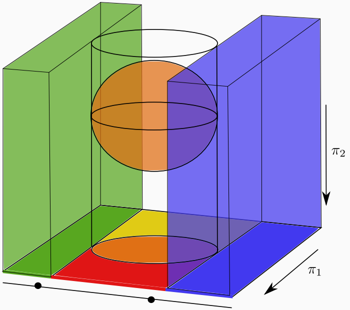

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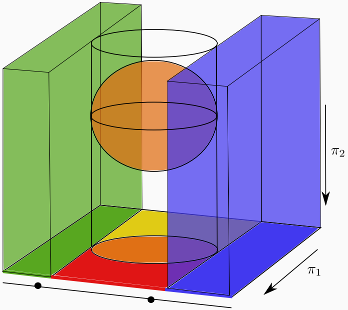
High complexity: $(sD)^{2^{O(n)}}$



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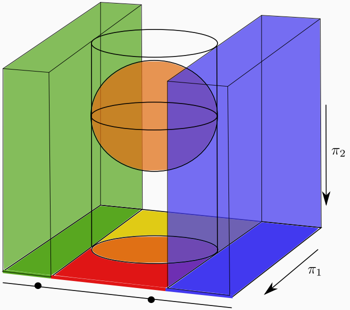

Oleinik-Petrovsky-Thom-Milnor's bound [Gabrielov & Vorobjov, 2009]

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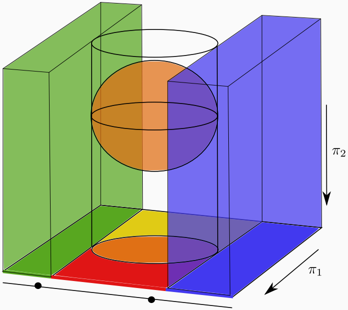

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Change of paradigm

$\rightsquigarrow$ Target <u>specific</u> problems:

e.g. solve connectivity queries

Contributions

Robotics applications

⇒ First **cuspidality** decision algorithm with singly exponential bit-complexity

- Roadmap computation for a challenging robotics problem

Computational real algebraic geometry can solve actual problems in robotics

Improve connectivity queries solving

- Nearly optimal roadmap algorithm for unbounded algebraic sets
- Efficient algorithm for connectivity of real algebraic curves

We have efficient algorithms for analyzing connectivity of real algebraic sets

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Cuspidality decision algorithm

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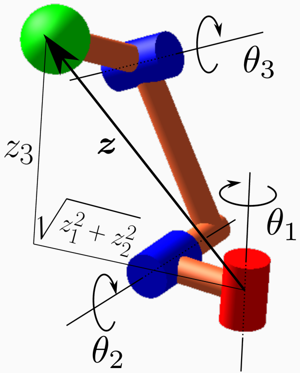
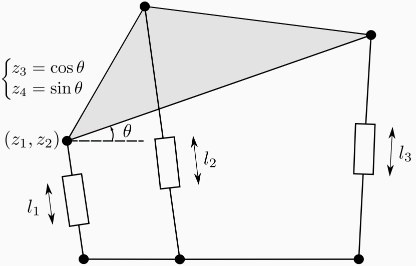


An Orthogonal 3R Serial Robot



A 3-RPR Planar Parallel Robot

A quick look at robotics

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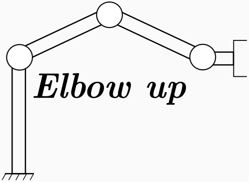
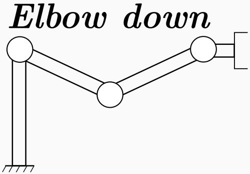


Associated postures

Two joint configurations $(\boldsymbol{\ell}, \boldsymbol{\theta})$ and $(\boldsymbol{\ell}', \boldsymbol{\theta}')$ s.t.

$$\mathcal{K}(\boldsymbol{\ell}, \boldsymbol{\theta}) = \mathcal{K}(\boldsymbol{\ell}', \boldsymbol{\theta}')$$



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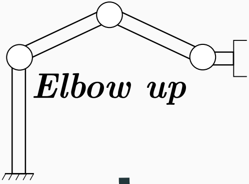

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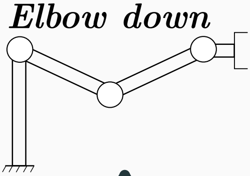

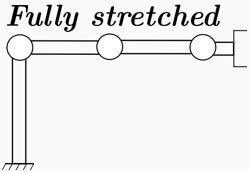

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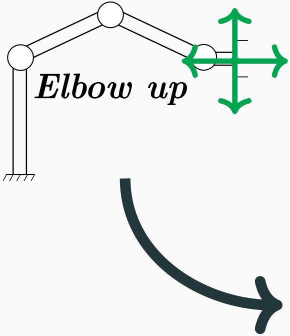

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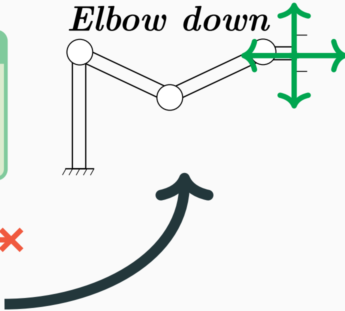

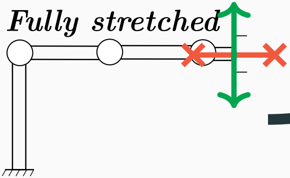

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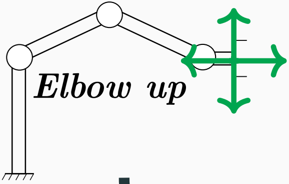

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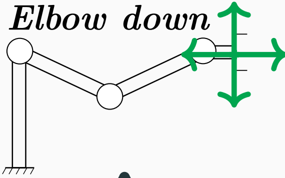

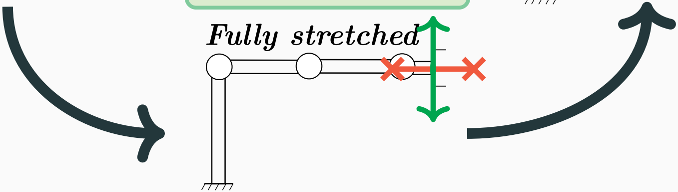

Singular posture

Configurations $(\boldsymbol{\ell}, \boldsymbol{\theta})$ s.t. $\mathrm{Jac}_{\boldsymbol{\ell}, \boldsymbol{\theta}}(\mathcal{K})$ is rank deficient

Cuspidal robot

Theorem [Borrel & Liégeois, 1986]

A robot **cannot** move between two associated postures, without passing by a singular posture

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Cuspidal robot [Wenger, 1992]

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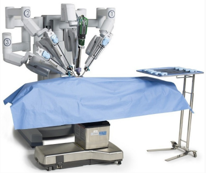
Motivation

Cuspidal robots can **induce problem** for task planning

Open problem

Cuspidality decision for a general robot



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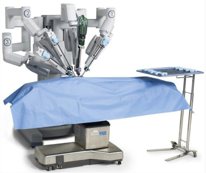
Motivation

Cuspidal robots can **induce problem** for task planning

Open problem

Cuspidality decision for a general robot



Contribution NEW!

First general algorithm

Cuspidal robot

~~Theorem~~ [Borrel & Liégois, 1986]

A robot **cannot** move between two associated postures,
without passing by a singular posture

[Wenger, 1992] → **WRONG !**

Cuspidal robot [Wenger, 1992]

Cuspidal robots **can** move between two associated postures,
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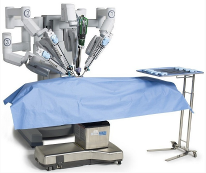
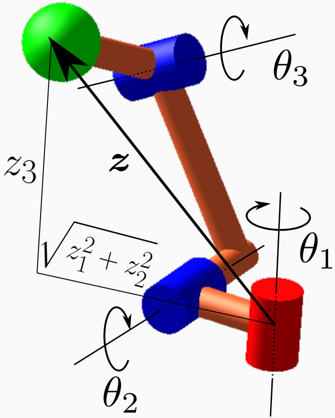


Contribution NEW!

First general algorithm with singly exponential complexity

Kinematic map

$$\begin{array}{cccc} \mathcal{K}\colon & \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \\ & (\boldsymbol{\ell}, \boldsymbol{\theta}) & \longmapsto & \boldsymbol{z}(\boldsymbol{\ell}, \boldsymbol{\theta}) \end{array}$$

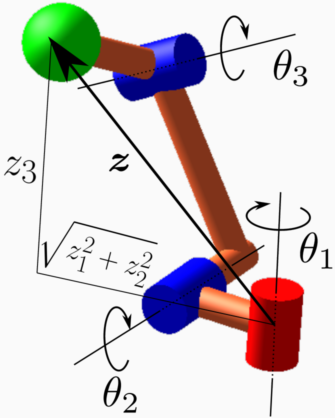


An algebro-geometric point of view

Kinematic map

$$\mathcal{K}\colon \quad \mathbb{R}^d \longrightarrow \mathbb{R}^d$$
$$(\boldsymbol{\ell}, \boldsymbol{\theta}) \longmapsto \boldsymbol{z}(\boldsymbol{\ell}, \boldsymbol{\theta})$$

$\mathcal{K}$ polynomial in $\boldsymbol{\ell}$, $c_j = \cos\theta_j$ and $s_j = \sin\theta_j$



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$\Uparrow$

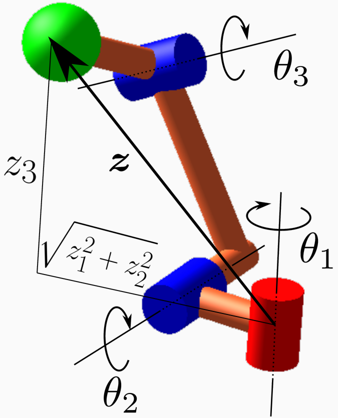
Change of variables:

$$r_i(\boldsymbol{\ell}, \boldsymbol{c}, \boldsymbol{s}) = z_i(\boldsymbol{\ell}, \boldsymbol{\theta})$$

with constraints

$$f_j(\boldsymbol{c}, \boldsymbol{s}) = {c_j}^2 + {s_j}^2 - 1 = 0$$

$\Downarrow$



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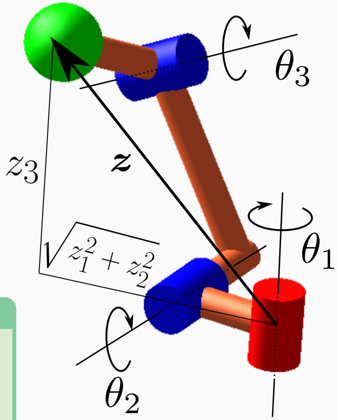
$\Downarrow$

Algebraic kinematic map

$$\widetilde{\mathcal{R}}\colon \begin{array}{ccc} \boldsymbol{V}(\boldsymbol{f}) \cap \mathbb{R}^n & \longrightarrow & \mathbb{R}^d \\ (\boldsymbol{\ell}, \boldsymbol{c}, \boldsymbol{s}) & \longmapsto & \big(r_1(\boldsymbol{\ell}, \boldsymbol{c}, \boldsymbol{s}), \ldots, r_d(\boldsymbol{\ell}, \boldsymbol{c}, \boldsymbol{s})\big) \end{array}$$

$\mathcal{R} = (r_1, \ldots, r_d)$ and $\boldsymbol{f} = (f_1, \ldots, f_s)$ in $\mathbb{R}[x_1, \ldots, x_n]$



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$$\mathrm{singP}(\mathcal{K}) = \left\{ (\boldsymbol{\ell}, \boldsymbol{\theta}) \mid \mathrm{Jac}_{\boldsymbol{\ell},\boldsymbol{\theta}}\, \mathcal{K} \text{ is rank deficient} \right\}$$

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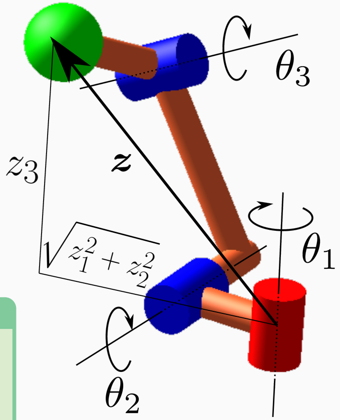
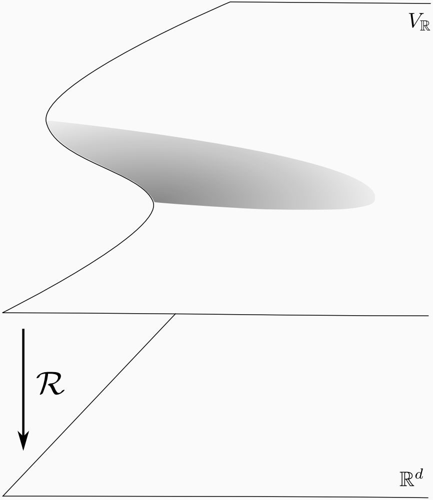


The algebraic cuspidality problem

Data

Data: $\boldsymbol{f} = (f_1, \ldots, f_s)$ and $\mathcal{R} = (r_1, \ldots, r_d)$ polynomials in $\mathbb{R}[x_1, \ldots, x_n]$

Assumptions: $V = \boldsymbol{V}(\boldsymbol{f})$ is d-equidimensional and $V_{\mathbb{R}} = V \cap \mathbb{R}^n \subsetneq \operatorname{sing}(V)$



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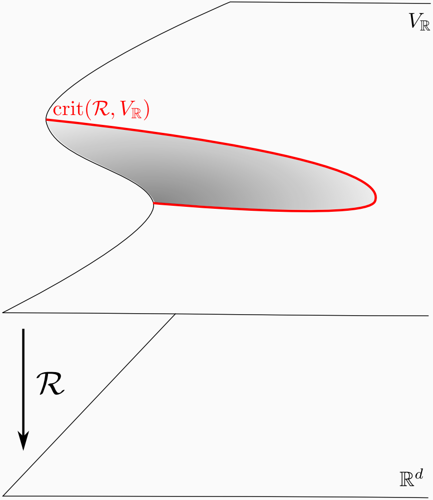

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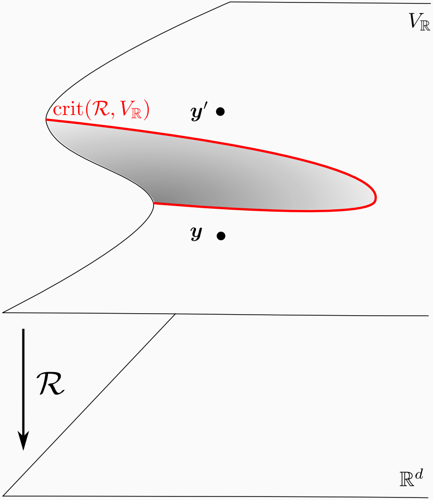
The restriction of $\mathcal{R}$ to $V_\mathbb{R}$ is cuspidal if there is $\boldsymbol{y} \neq \boldsymbol{y}' \in V_\mathbb{R}$ such that



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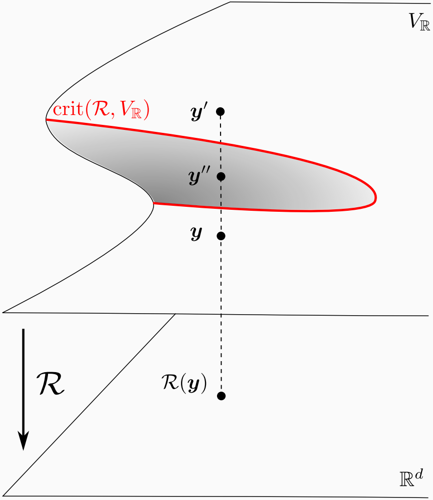

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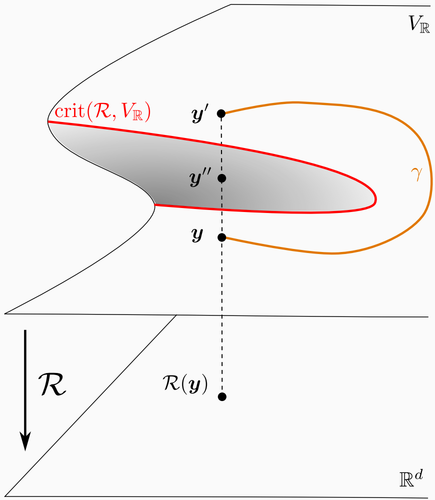

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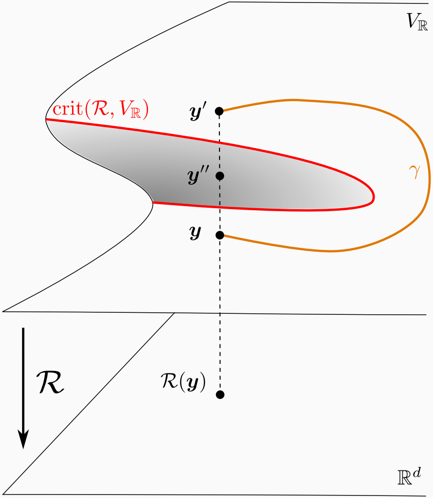

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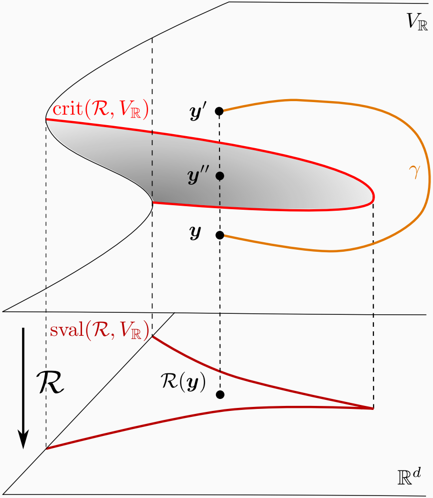
Singular values of $\mathcal{R}$

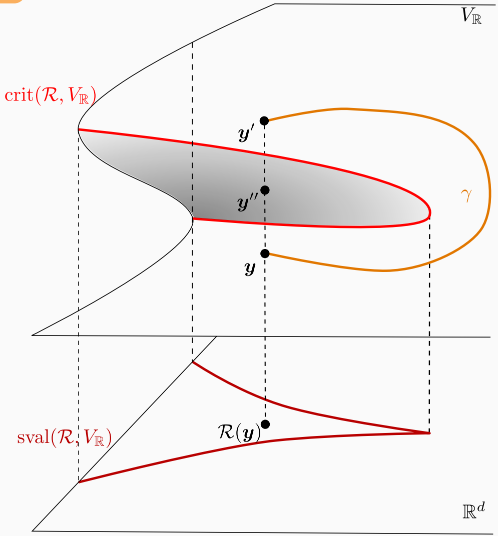
$\operatorname{sval}(\mathcal{R}, V) = \mathcal{R}\big(\operatorname{crit}(\mathcal{R}, V)\big)$



The cuspidality algorithm

Thom's First Isotopy Lemma

Fibers from the same connected component of $\mathbb{R}^d - \mathrm{sval}(\mathcal{R}, V)$ have the **same type**



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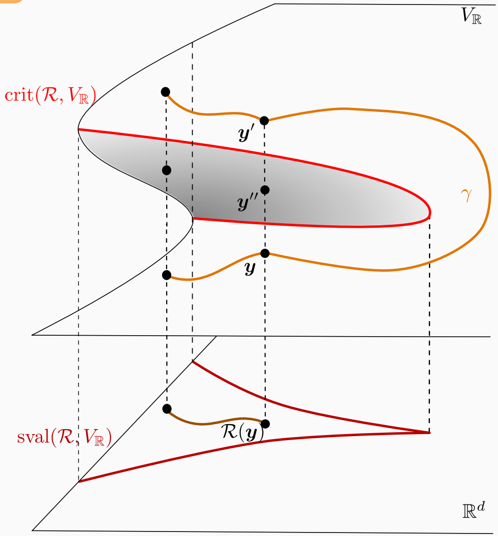

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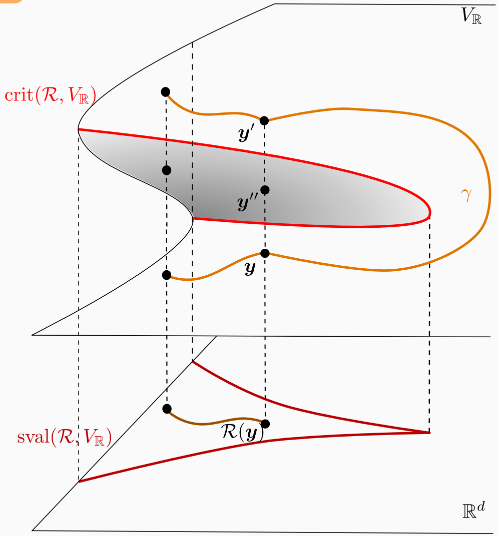

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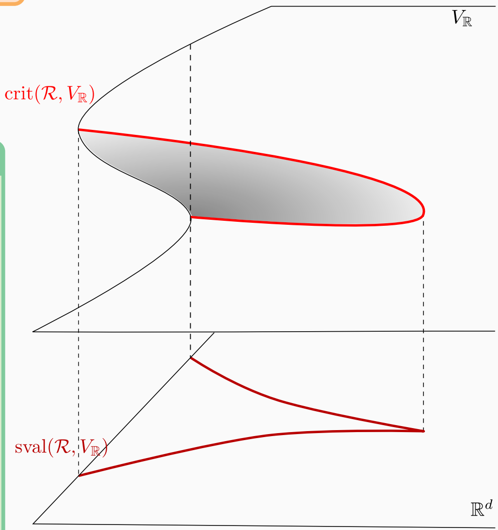

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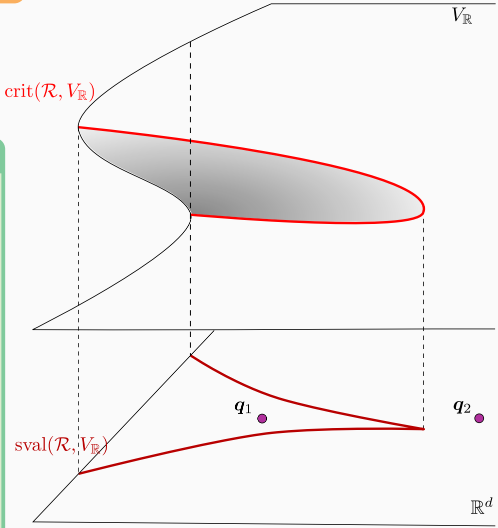

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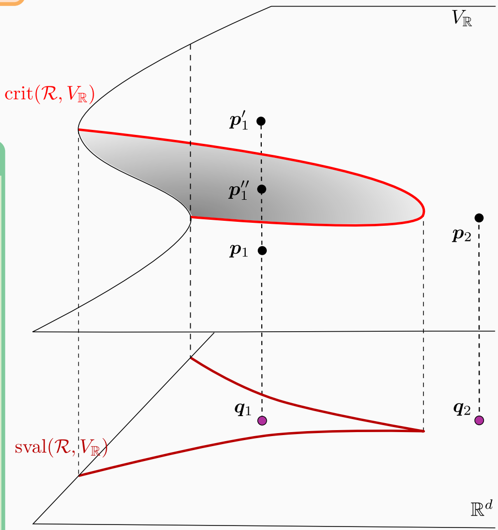

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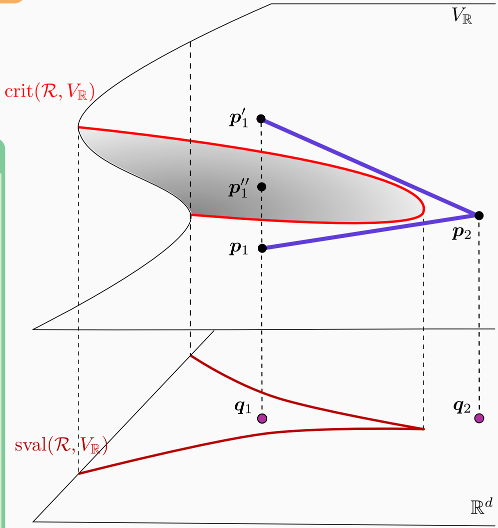

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Algebraic set

$V = \boldsymbol{V}(\boldsymbol{f}) \subset \mathbb{C}^n$
$\dim(V) = d$

Soft-O notation

$\tilde{O}(N) = O(N \log^a N)$

Magnitude

$\mathrm{degrees}(\boldsymbol{f}) \leq D$ and $|\mathrm{coeffs}(\boldsymbol{f})| \leq 2^{\tau}$

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Projection

$\tau(nD)^{O(nd)}$

[Basu & Pollack & Roy, '16]

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Connectivity queries

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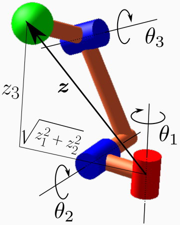

Implementation

Prototype applied to two 3R robots

Main steps

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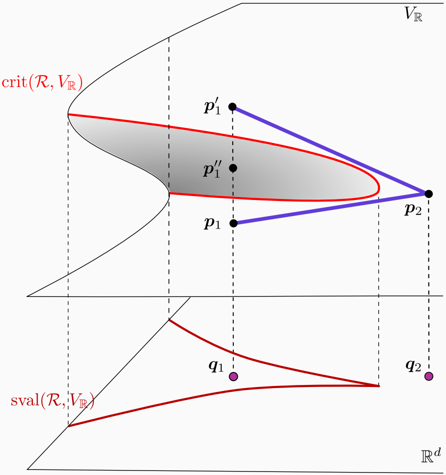
3. Compute their preimages $\mathcal{P} = V \cap \mathcal{R}^{-1}(\mathcal{Q})$
4. Search for cuspidal pairs in $\mathcal{P}$ by connecting points in the same connected component of $V_{\mathbb{R}} - \mathrm{crit}(\mathcal{R}, V)$



Connectivity queries

$\tilde{O}(\tau)(nD)^{O(n^2)}$

[Basu & Pollack & Roy, '00]

Contributions

Robotics applications

✓ First **cuspidality** decision algorithm with singly exponential bit-complexity

∘ Roadmap computation for a challenging robotics problem

Computational real algebraic geometry can solve actual problems in robotics

Improve connectivity queries solving

⇒ Nearly optimal **roadmap** algorithm for unbounded algebraic sets

∘ Efficient algorithm for connectivity of real algebraic curves

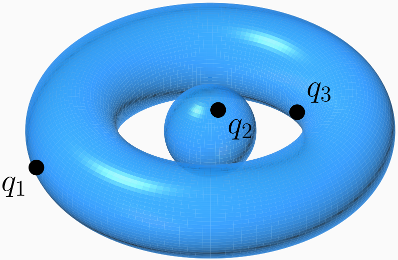
We have efficient algorithms for analyzing connectivity of real algebraic sets

Computing connectivity properties: Roadmaps

[Canny, 1988] Compute $\mathscr{R} \subset S$ one-dimensional, sharing its connectivity

Roadmap of $(S, \mathcal{P})$

A semi-algebraic curve $\mathscr{R} \subset S$, containing query points $(q_1, \ldots, q_N)$ s.t.
for all connected components C of S: $C \cap \mathscr{R}$ is non-empty and connected



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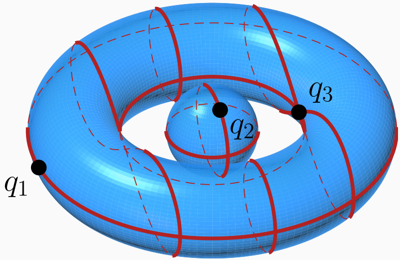

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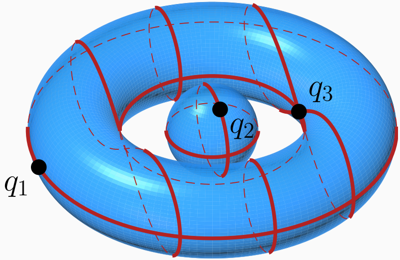
Proposition

q_i and q_j are path-connected in $S \iff$ they are in $\mathscr{R}$

Problem reduction

Arbitrary dimension



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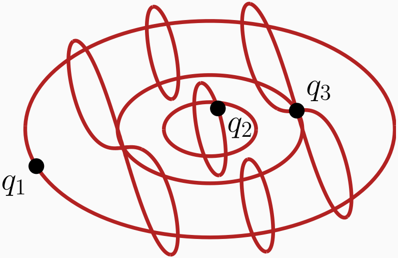
Problem reduction

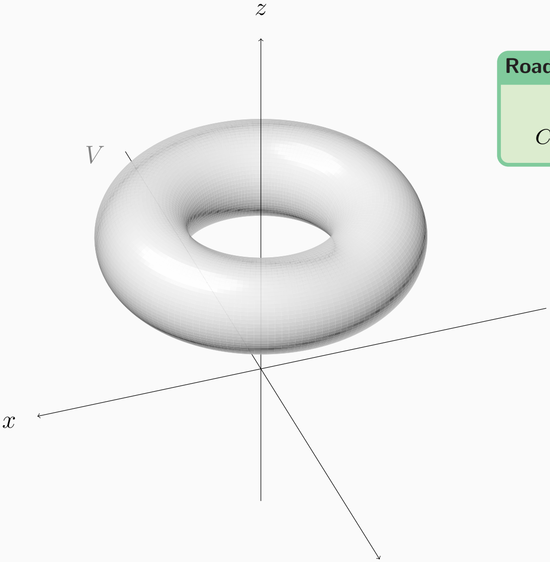
Arbitrary dimension $\underset{\text{ROADMAP}}{\Longrightarrow}$ Dimension 1



Roadmap algorithms for unbounded algebraic sets

joint work with M. Safey El Din and É. Schost

Canny's strategy



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$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

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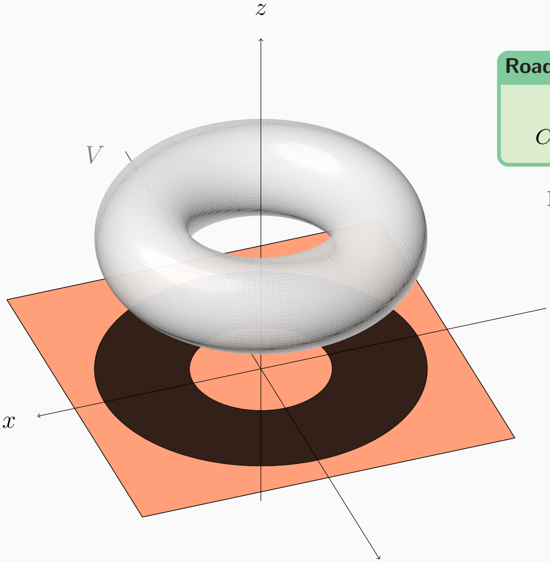

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Projection through:
$\pi_2 \colon (x_1, \dots, x_n) \mapsto (x_1, x_2)$

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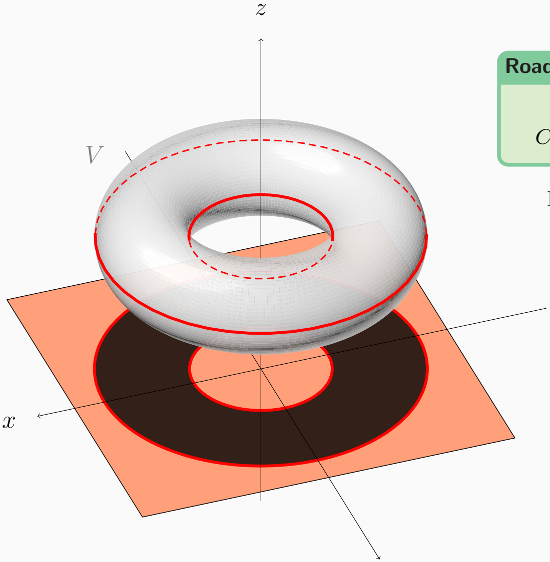

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$W(\pi_2, V)$ critical locus of π_2.

Intersects all the
connected components of V

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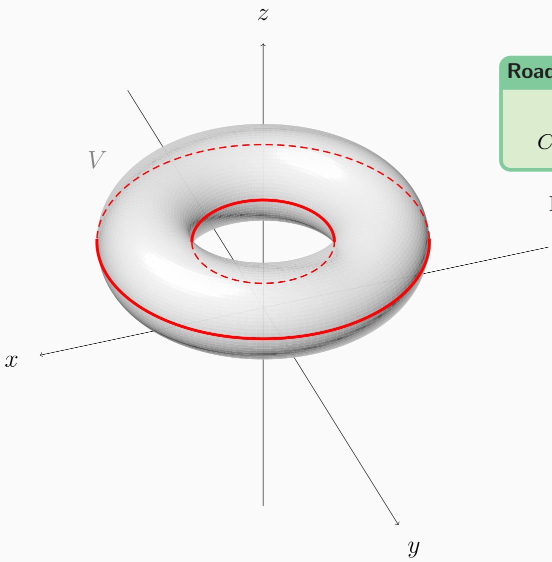

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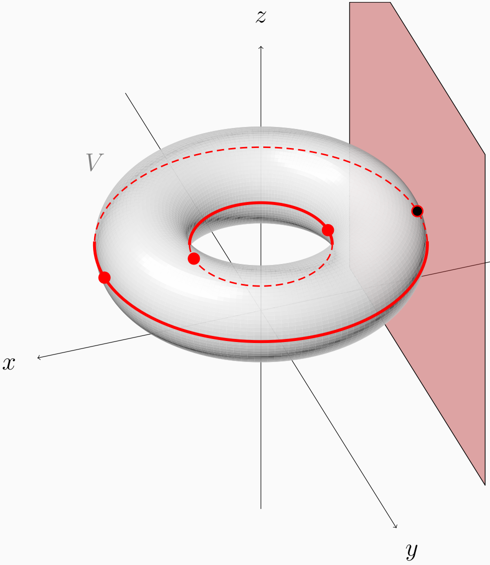

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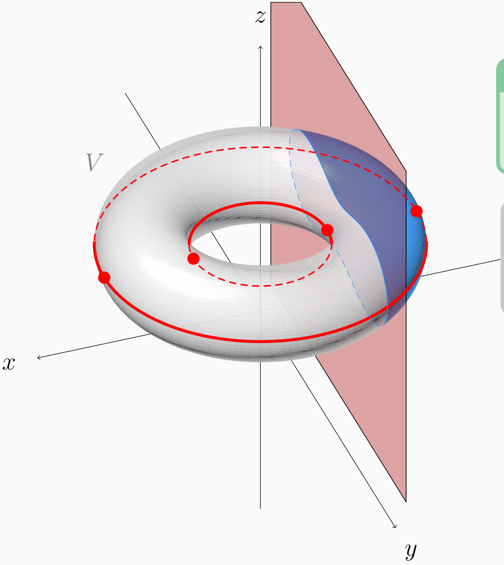
Morse theory

"Scan" $\boldsymbol{W(\pi_2, V)}$ at the critical values of π_1

- We repair the connectivity failures with critical fibers
- We repeat the process at every critical value

Canny's strategy



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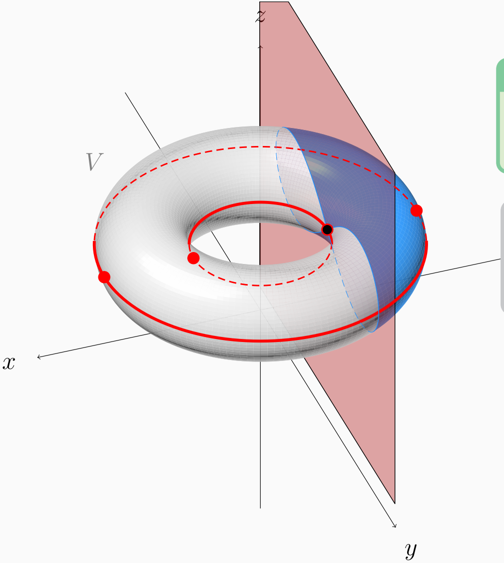

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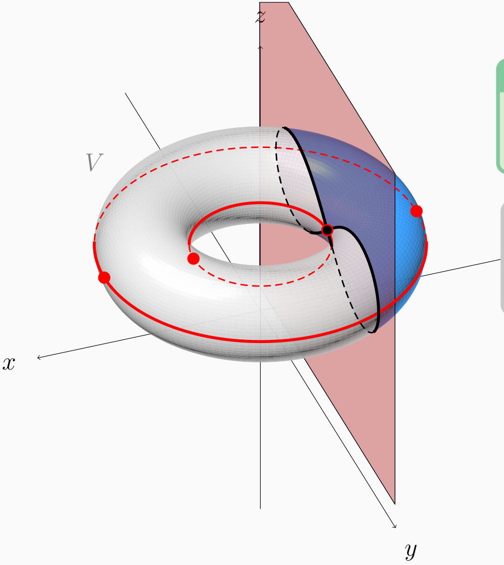

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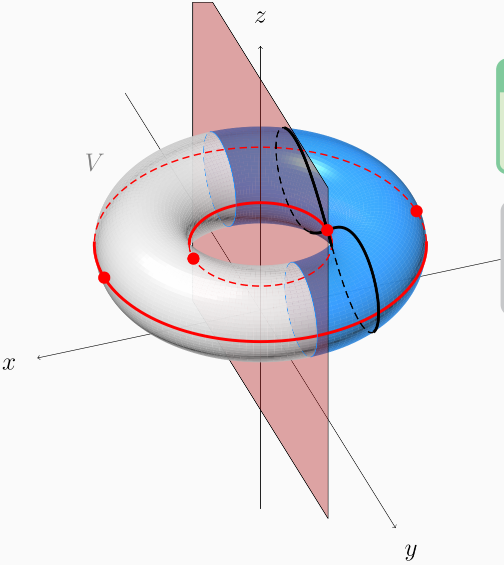

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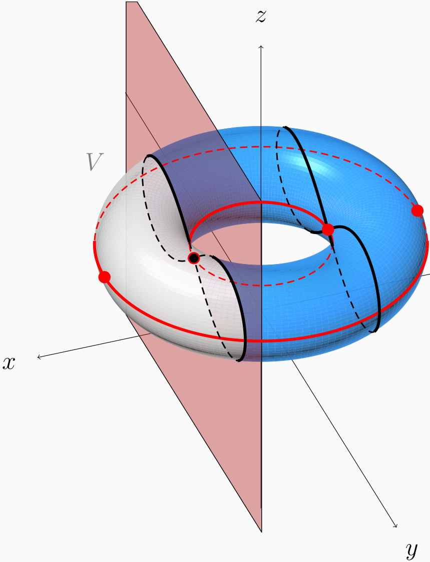

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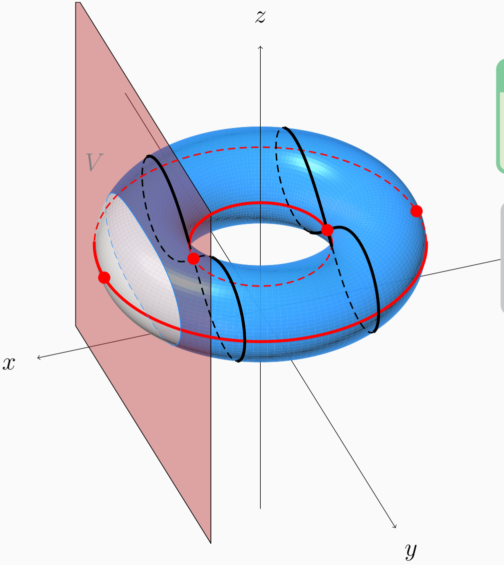

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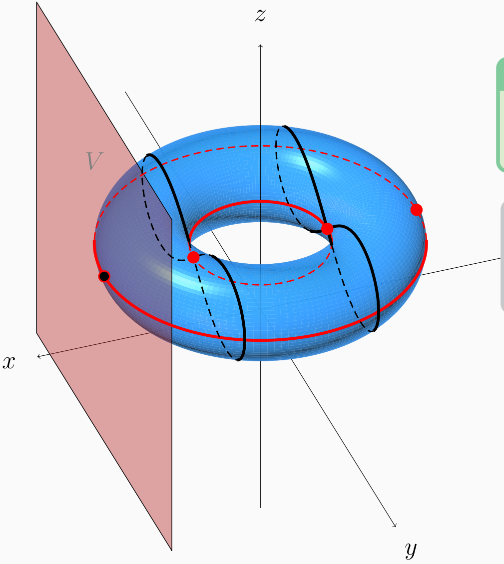

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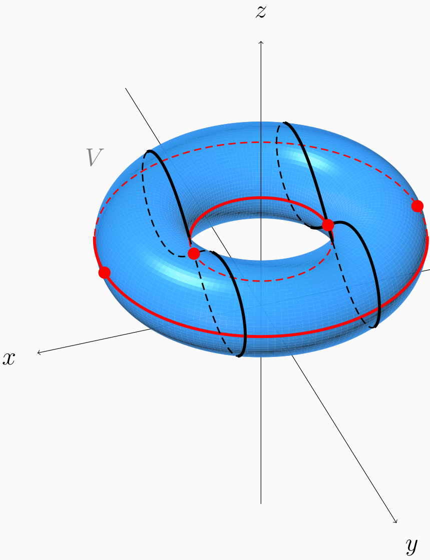

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$W(\pi_2, V)$ polar variety
F critical fibers

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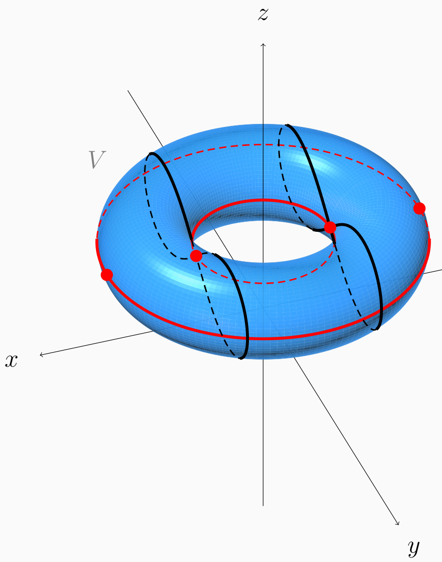

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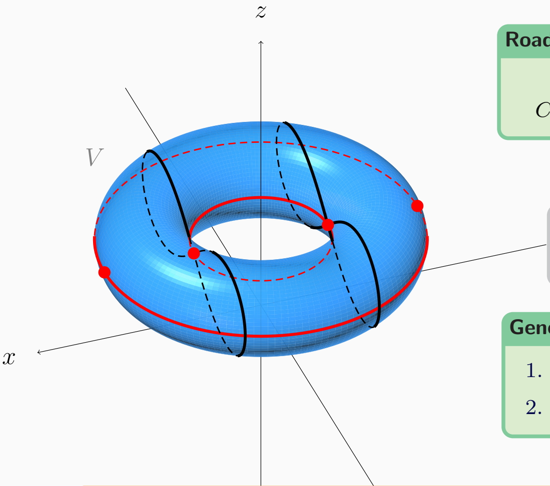
Genericity assumptions

1. $W(\pi_2, V)$ has dimension 1
2. F has dimension $\dim(V) - 1$

Canny's strategy



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Theorem [Canny, 1988]

If V is bounded, $W(\pi_2, V) \bigcup F$ has dimension $\dim(V) - 1$
and satisfies the Roadmap property

On the complexity of computing roadmaps

$S \subset \mathbb{R}^n$ semi alg. set of dimension d and defined by s polynomials of degree $\leqslant D$

Connectivity result [Canny, 1988]

If V is bounded, $W(\pi_2, V) \cup F$ has dimension $d-1$
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Author·s	Complexity	Assumptions
[Schwartz & Sharir, 1983]	$(sD)^{2^{O(n)}}$	

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Connectivity result [Safey El Din & Schost, 2011]

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Remove the boundedness assumption is a *costly* step

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Not polynomial in the output size

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Necessity of a new theorem in the **unbounded** case!

On the extension of Canny's result

Projection on 2 coordinates

$$\pi_2 : \begin{array}{ccc} \mathbb{C}^n & \to & \mathbb{C}^2 \\ (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) & \mapsto & (\boldsymbol{x}_1, \boldsymbol{x}_2) \end{array}$$

- $W(\pi_2, V)$ polar variety
- $F_2 = \pi_1^{-1}(\pi_1(K)) \cap V$ critical fibers
- K = critical points of π_1 on $W(\pi_2, V)$

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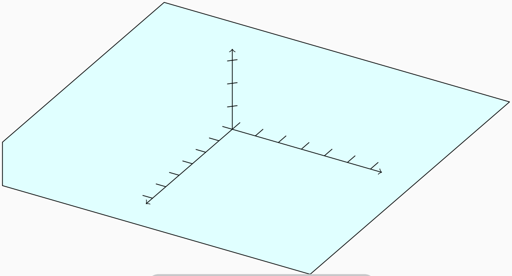
- $K =$ critical points of π_1 on $W(\pi_i, V)$

Connectivity result [Safey El Din & Schost, 2011]

If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$ and satisfies the Roadmap property



No critical points...

On the extension of Canny's result

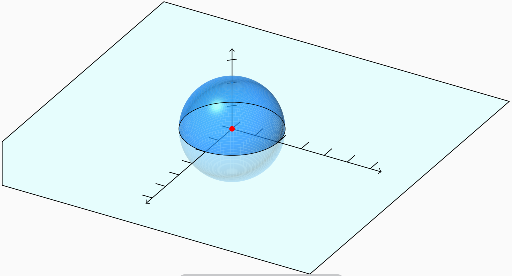
Non-negative proper polynomial map

$$\begin{array}{rccc} \boldsymbol{\varphi}_i: & \mathbb{C}^n & \longrightarrow & \mathbb{C}^i \\ & \boldsymbol{x} & \mapsto & (\psi_1(\boldsymbol{x}), \dots, \psi_i(\boldsymbol{x})) \end{array}$$

- $W(\boldsymbol{\varphi}_i, V)$ generalized polar variety
- $F_i = \boldsymbol{\varphi}_{i-1}^{-1}(\boldsymbol{\varphi}_{i-1}(K)) \cap V$ critical fibers.
- $K =$ critical points of $\boldsymbol{\varphi}_1$ on $W(\boldsymbol{\varphi}_i, V)$

Connectivity result [P. & Safey El Din & Schost, 2024] NEW!

~~If V is bounded,~~ $W(\boldsymbol{\varphi}_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$
and satisfies the Roadmap property



Critical point!

- ⇝ Sard's lemma
- ⇝ Thom's isotopy lemma
- ⇝ Puiseux series

How to use it?

Assumptions to satisfy in the new result

(R) $\mathrm{sing}(V)$ is finite

(P) $\boldsymbol{\varphi}_1$ is a proper map bounded from below

For all $1 \leqslant i \leqslant \dim(V)/2$,

(N) $\boldsymbol{\varphi}_{i-1}$ has finite fibers on W_i

(W) $\dim W_i = i - 1$ and $\mathrm{sing}(W_i) \subset \mathrm{sing}(V)$

(F) $\dim F_i = n - d + 1$ and $\mathrm{sing}(F_i)$ is finite

How to use it?

Assumptions to satisfy in the new result

(R) $\mathrm{sing}(V)$ is finite ✓

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Assumption on the input

How to use it?

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By construction of φ

A successful candidate

Choose *generic* $(\boldsymbol{a}, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n) \in \mathbb{R}^{n^2}$ and:

$$\boldsymbol{\varphi} = \left(\sum_{i=1}^{n} (x_i - a_i)^2 \, , \, \boldsymbol{b}_2^{\mathsf{T}} \vec{\boldsymbol{x}} \, , \, \ldots \, , \, \boldsymbol{b}_n^{\mathsf{T}} \vec{\boldsymbol{x}} \right) \quad \text{where} \quad a_i \in \mathbb{R}, \quad \boldsymbol{b}_i \in \mathbb{R}^n$$

It satisfies the assumptions! NEW!

How to use it?

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Generalization of Noether position from [Safey El Din & Schost, 2003]

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Jacobian criterion

$\oplus$

Thom's transversality theorem

A successful candidate

Choose *generic* $(\boldsymbol{a}, \boldsymbol{b}_2, \dots, \boldsymbol{b}_n) \in \mathbb{R}^{n^2}$ and:

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It satisfies the assumptions! NEW!

How to use it?

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Noether position

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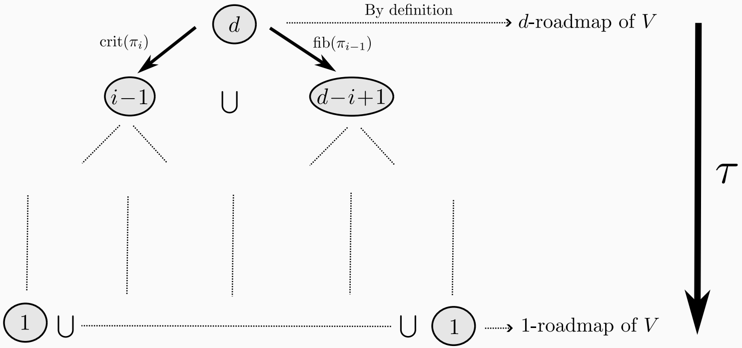
It satisfies the assumptions! NEW!

An algorithm for unbounded algebraic set

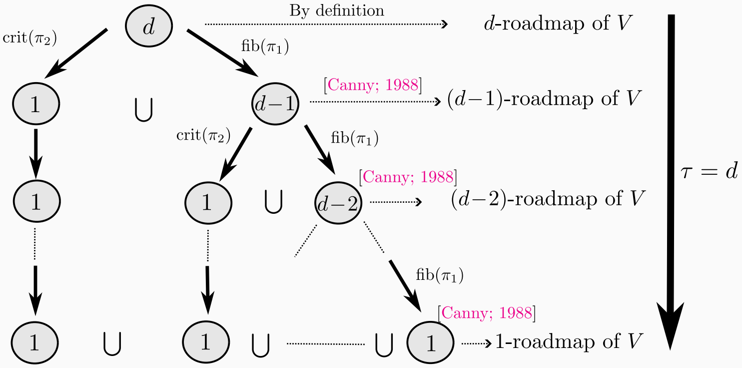
Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d



Depth of recursion tree : τ

$\Rightarrow$ complexity: $(nD)^{O(n\tau)}$

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d



Depth of recursion tree : d

$\Rightarrow$ complexity: $(nD)^{O(nd)}$

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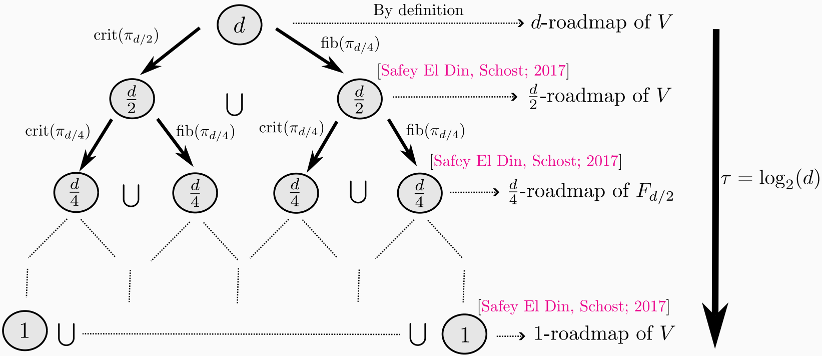

Depth of recursion tree : $\log_2(d)$

$\Rightarrow$ complexity: $(nD)^{O(n \log_2(d))}$

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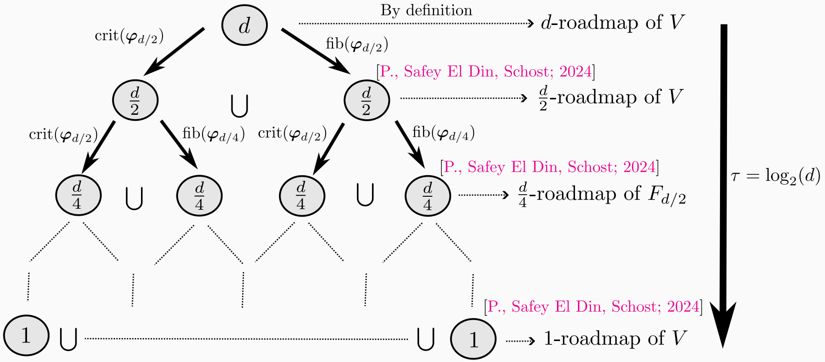

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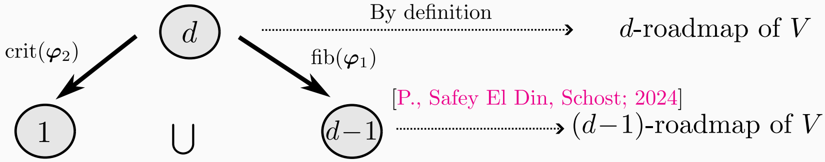

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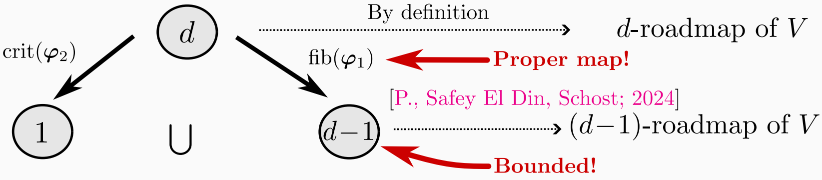

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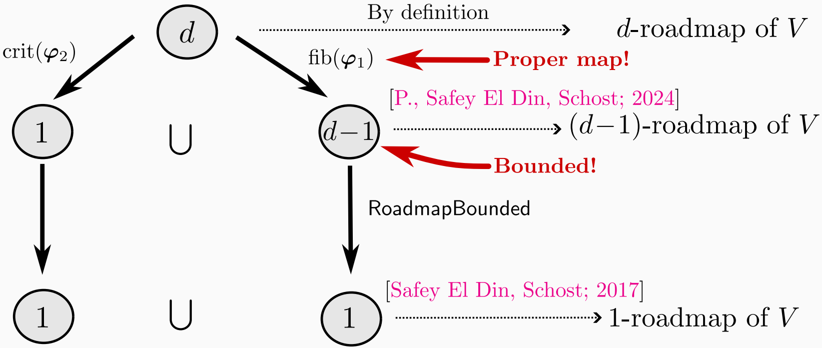

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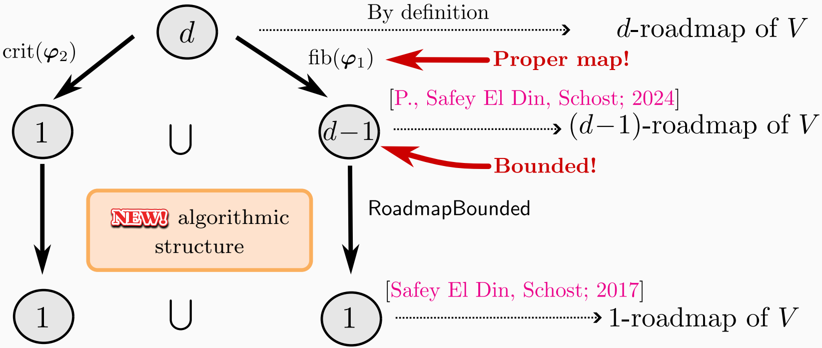
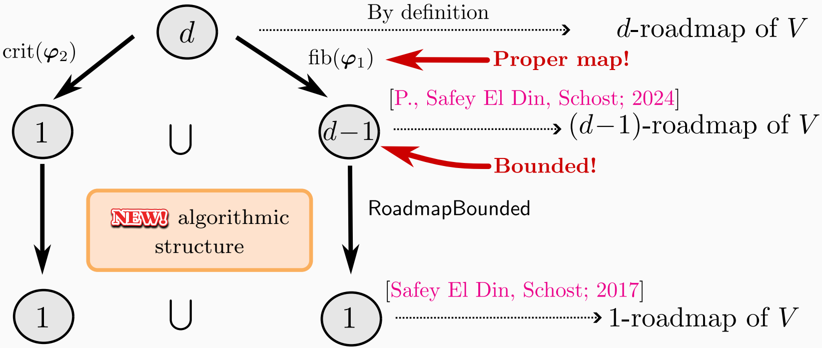


Quantitative estimate

	Output size	Complexity
RoadmapBounded(fib(φ_1)) Compute crit(φ_2) & fib(φ_1)		
Overall		

An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d

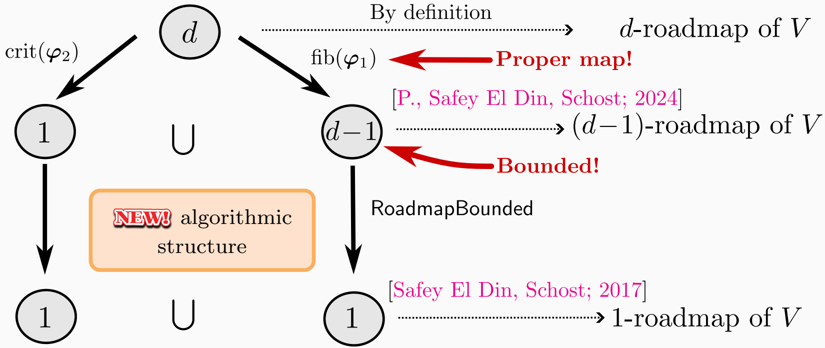


Quantitative estimate

	Output size	Complexity
RoadmapBounded$(\mathrm{fib}(\boldsymbol{\varphi}_1))$ Compute $\mathrm{crit}(\boldsymbol{\varphi}_2)$ & $\mathrm{fib}(\boldsymbol{\varphi}_1)$	$(n^2 D)^{4n \log_2 d + O(n)}$	$(n^2 D)^{6n \log_2 d + O(n)}$
Overall		

An algorithm for unbounded algebraic set

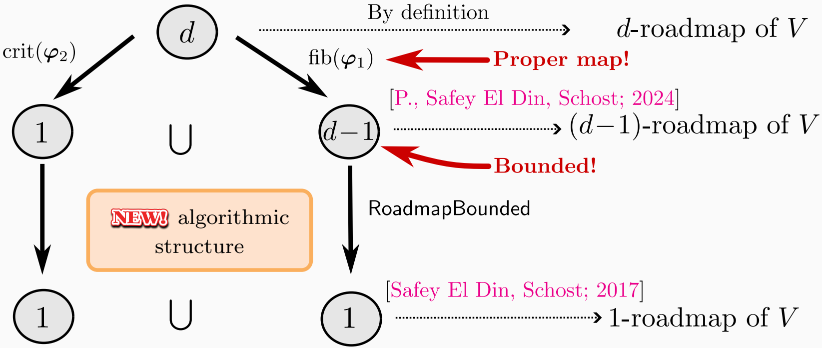
Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d



Quantitative estimate

	Output size	Complexity
RoadmapBounded(fib(φ_1))	$(n^2D)^{4n\log_2 d+O(n)}$	$(n^2D)^{6n\log_2 d+O(n)}$
Compute crit(φ_2) & fib(φ_1)	$(nD)^{O(n)}$	$(nD)^{O(n)}$
Overall		

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d



Quantitative estimate

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Compute crit(φ_2) & fib(φ_1)	$(nD)^{O(n)}$	$(nD)^{O(n)}$
Overall	$(n^2D)^{4n\log_2 d+O(n)}$	$(n^2D)^{6n\log_2 d+O(n)}$

Summary

Input

Polynomials in $\mathbb{Q}[x_1, \ldots x_n]$ of max degree D defining a smooth algebraic set of dim. d

Connectivity reduction process - before

Arbitrary dimension $\overset{\text{Roadmap}}{\Longrightarrow}$ Dimension: 1 $\overset{\text{Topology}}{\Longrightarrow}$ Finite graph $\mathscr{G}$

$\downarrow$ $\downarrow$

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Arbitrary dimension $\xrightarrow{\text{Roadmap}}$ Dimension: 1, Size: $(nD)^{O(n \log(n))}$ $\xrightarrow{\text{Topology}}$ Finite graph $\mathscr{G}$

$\downarrow$ $(nD)^{O(n \log^2(n))}$ [Basu, Roy; 2014]

$\downarrow$

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ROADMAP ↓ $(nD)^{O(n\log^2(n))}$ [Basu, Roy; 2014]

Topology ↓ $(\text{Size})^{O(1)} = (nD)^{O(n\log(n))}$ [Safey El Din, Schost; 2011]

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Connectivity reduction process - now

Arbitrary dimension $\xrightarrow{\text{ROADMAP}}$ Dimension: 1 $\Longrightarrow$ Finite graph $\mathscr{G}$

$\downarrow$

$\downarrow$

Summary

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Arbitrary dimension $\xrightarrow{\text{ROADMAP}}$ Dimension: 1, Size: $(nD)^{O(n\log(n))}$ $\xrightarrow{\text{Topology}}$ Finite graph $\mathscr{G}$

↓ $(nD)^{O(n\log^2(n))}$ [Basu, Roy; 2014]

↓ $(\mathrm{Size})^{O(1)} = (nD)^{O(n\log(n))}$ [Safey El Din, Schost; 2011]

Connectivity reduction process - now

Arbitrary dimension $\xrightarrow{\text{ROADMAP}}$ Dimension: 1, Size: $(n^2D)^{4n\log_2(d)+O(n)}$ $\Longrightarrow$ Finite graph $\mathscr{G}$

↓ $(n^2D)^{6n\log_2(d)+O(n)}$ [P., Safey El Din, Schost; 2024]

↓

Summary

Input

Polynomials in $\mathbb{Q}[x_1, \dots x_n]$ of max degree D defining a smooth algebraic set of dim. d

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Computing roadmaps in unbounded smooth real algebraic sets I: connectivity results, 2024 with M. Safey El Din and É. Schost

Computing roadmaps in unbounded smooth real algebraic sets II: algorithm and complexity, 2024 with M. Safey El Din and É. Schost

Contributions

Robotics applications

- ✓ First **cuspidality** decision algorithm with singly exponential bit-complexity
- ⇒ Roadmap **computation** for a challenging robotics problem

Computational real algebraic geometry can solve actual problems in robotics

Improve connectivity queries solving

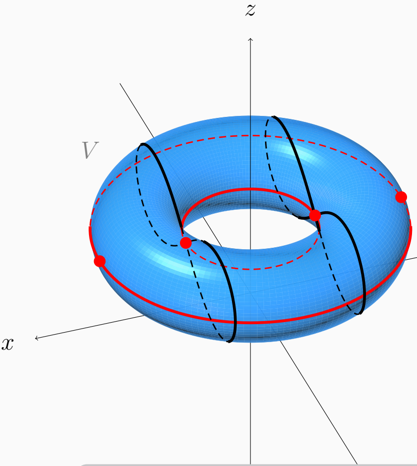
- ✓ Nearly optimal **roadmap** algorithm for unbounded algebraic sets
  ⇝ Complexity: $(n^2D)^{6n\log_2 d+O(n)}$ ⇝ Output size: $(n^2D)^{4n\log_2 d+O(n)}$
- ○ Efficient algorithm for connectivity of real algebraic curves

We have efficient algorithms for analyzing connectivity of real algebraic sets

Analysis of the kinematic singularities of a PUMA robot

with J.Capco, M.Safey El Din and P.Wenger

Canny's strategy



Roadmap property

$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

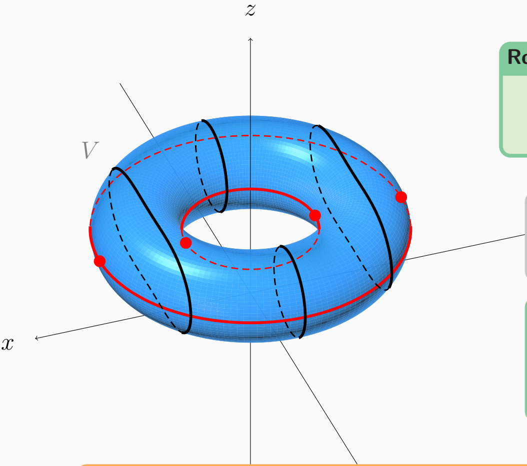
$W(\pi_2, V)$ polar variety
F critical fibers

Genericity assumptions

1. $W(\pi_2, V)$ has dimension 1
2. F has dimension $\dim(V) - 1$

Theorem [Canny, 1988]

If V is bounded, $W(\pi_2, V) \bigcup F$ has dimension $\dim(V) - 1$
and satisfies the Roadmap property



Roadmap property

$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

$\boldsymbol{W(\pi_2, V)}$ **polar variety**
$\boldsymbol{F}$ **regular fibers**

Genericity assumptions

1. $\boldsymbol{W(\pi_2, V)}$ has dimension 1
2. $\boldsymbol{F}$ has dimension $\dim(V) - 1$

Theorem [Mezzaroba & Safey El Din, 2006]

If V is bounded, $\boldsymbol{W(\pi_2, V)} \bigcup \boldsymbol{F}$ has dimension $\dim(V) - 1$
and satisfies the Roadmap property

Roadmap computation for robotics

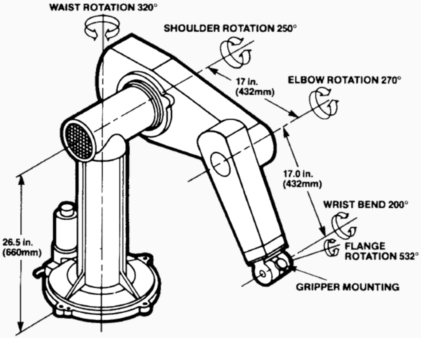
Matrix M associated to a PUMA-type robot with a non-zero offset in the wrist

$$\begin{bmatrix} (v_3+v_2)(1-v_2v_3) & 0 & A(\boldsymbol{v}) & d_3A(\boldsymbol{v}) & a_2(v_3^2+1)(v_2^2-1)-a_3A(\boldsymbol{v}) & 2d_3(v_3+v_2)(v_2v_3-1) \\ 0 & v_3^2+1 & 0 & 2a_2v_3 & 0 & (a_3-a_2)v_3^2+a_2+2a_3 \\ 0 & 1 & 0 & 0 & 0 & 2a_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ v_4 & 1-v_4^2 & 0 & d_4(1-v_4^2) & -2d_4v_4 & 0 \\ (v_4^2-1)v_5 & 4v_4v_5 & (1-v_5^2)(v_4^2+1) & (1-v_5^2)(v_4^2-1)d_5+4d_4v_4v_5 & 2d_5v_4(1-v_5^2)+2d_4v_5(1-v_4^2) & -2d_5v_5(v_4^2+1) \end{bmatrix}$$

$$S = \{\boldsymbol{v} \in \mathbb{R}^4 \mid \det(M(\boldsymbol{v})) \neq 0\}$$

https://msolve.lip6.fr

- ⇝ Multivariate system solving
- ⇝ Real roots isolation



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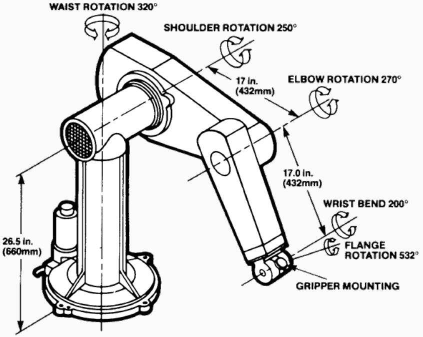
$$S = \{\boldsymbol{v} \in \mathbb{R}^4 \mid \det(M(\boldsymbol{v})) \neq 0\}$$

First step

Max. deg without splitting: **1858**

Locus	Degrees	$\mathbb{R}$-roots	Tot. time
Critical points	400 & 934	96 & 182	9.7 min
Critical curves	182 & 220	∞	3h46



A PUMA 560 [Unimation, 1984]

Roadmap computation for robotics

Matrix M associated to a PUMA-type robot with a non-zero offset in the wrist

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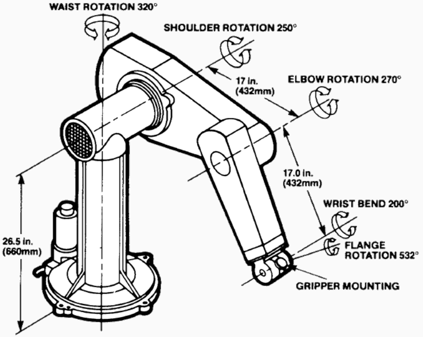

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Locus	Degrees	$\mathbb{R}$-roots	Tot. time
Critical points	400 & 934	96 & 182	9.7 min
Critical curves	182 & 220	∞	3h46

Recursive step over 95 fibers

Data are for one fiber

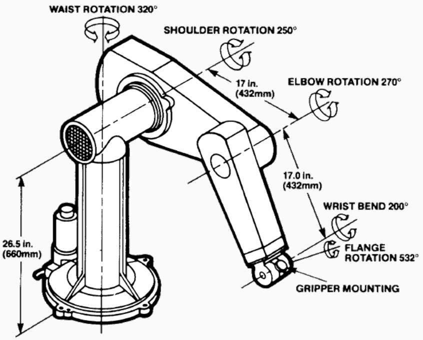
Locus	Degrees	$\mathbb{R}$-roots	Total time
Critical points	38	14	6.4 min
Critical curves	21	∞	9.6 min

Roadmap computation for robotics

Matrix M associated to a PUMA-type robot with a non-zero offset in the wrist

$$\begin{bmatrix} (v_3+v_2)(1-v_2v_3) & 0 & A(\boldsymbol{v}) & d_3A(\boldsymbol{v}) & a_2(v_3^2+1)(v_2^2-1)-a_3A(\boldsymbol{v}) & 2d_3(v_3+v_2)(v_2v_3-1) \\ 0 & v_3^2+1 & 0 & 2a_2v_3 & 0 & (a_3-a_2)v_3^2+a_2+2a_3 \\ 0 & 1 & 0 & 0 & 0 & 2a_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ v_4 & 1-v_4^2 & 0 & d_4(1-v_4^2) & -2d_4v_4 & 0 \\ (v_4^2-1)v_5 & 4v_4v_5 & (1-v_5^2)(v_4^2+1) & (1-v_5^2)(v_4^2-1)d_5+4d_4v_4v_5 & 2d_5v_4(1-v_5^2)+2d_4v_5(1-v_4^2) & -2d_5v_5(v_4^2+1) \end{bmatrix}$$

$$S = \{\boldsymbol{v} \in \mathbb{R}^4 \mid \det(M(\boldsymbol{v})) \neq 0\}$$



A PUMA 560 [Unimation, 1984]

First step

Max. deg without splitting: **1858**

Locus	Degrees	$\mathbb{R}$-roots	Tot. time
Critical points	400 & 934	96 & 182	9.7 min
Critical curves	182 & 220	∞	3h46

Recursive step over 95 fibers

Data are for one fiber

Locus	Degrees	$\mathbb{R}$-roots	Total time
Critical points	38	14	6.4 min
Critical curves	21	∞	9.6 min

Roadmap computation NEW!

Output degree: **4847**

Time: **4h10** (`msolve`)

Contributions

Robotics applications

✓ First **cuspidality** decision algorithm with singly exponential bit-complexity

✓ Roadmap **computation** for a challenging robotics problem

Computational real algebraic geometry can solve actual problems in robotics

Improve connectivity queries solving

✓ Nearly optimal **roadmap** algorithm for unbounded algebraic sets
$\rightsquigarrow$ Complexity: $(n^2D)^{6n\log_2 d+O(n)}$ $\rightsquigarrow$ Output size: $(n^2D)^{4n\log_2 d+O(n)}$

$\Rightarrow$ Efficient algorithm for connectivity of real algebraic **curves**

We have efficient algorithms for analyzing connectivity of real algebraic sets

Computing connectivity properties: Roadmaps

[Canny, 1988] Compute $\mathscr{R} \subset S$ one-dimensional, sharing its connectivity

Roadmap of $(S, \mathcal{P})$

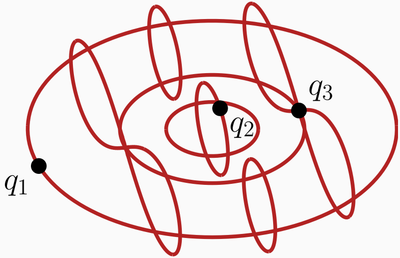
A *semi-algebraic curve* $\mathscr{R} \subset S$, containing query points $(q_1, \ldots, q_N)$ s.t. for all connected components C of S: $C \cap \mathscr{R}$ is *non-empty* and *connected*

Proposition

q_i and q_j are path-connected in $S \iff$ they are in $\mathscr{R}$

Problem reduction

Arbitrary dimension $\underset{\text{ROADMAP}}{\Longrightarrow}$ Dimension 1



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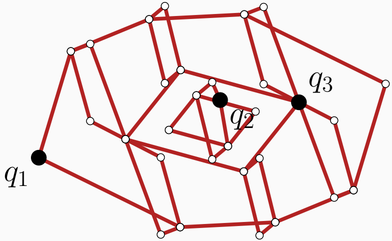
Problem reduction

Arbitrary dimension $\underset{\text{ROADMAP}}{\Longrightarrow}$ Dimension 1 $\underset{\text{Topology}}{\Longrightarrow}$ Finite graph $\mathscr{G}$



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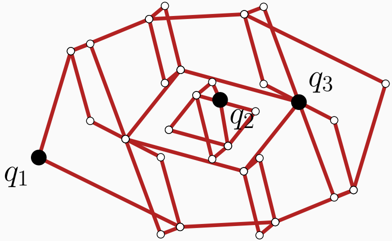

Algorithm for connectivity queries on real algebraic curves

joint work with Md N.Islam and A.Poteaux

Data representation and quantitative estimate

Theorem

In a *generic* system of coordinates,
V is *birational* to a hypersurface of $\mathbb{C}^{d+1}$ through:

$$\pi_{d+1} \colon (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \mapsto (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+1})$$

$\longleftarrow$ V equidimensional of dimension d

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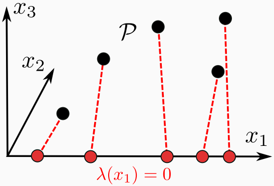
Zero-dimensional parametrization of $\mathcal{P} \subset \mathbb{C}^n$ finite

$(\lambda, \vartheta_2, \ldots, \vartheta_n) \subset \mathbb{Z}[x_1]$ s.t.

$$\mathcal{P} = \left\{ \left(\boldsymbol{x}_1, \frac{\vartheta_2(\boldsymbol{x}_1)}{\lambda'(\boldsymbol{x}_1)}, \ldots, \frac{\vartheta_n(\boldsymbol{x}_1)}{\lambda'(\boldsymbol{x}_1)} \right) \text{ s.t. } \lambda(\boldsymbol{x}_1) = 0 \right\}$$



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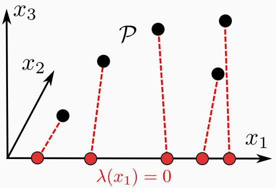
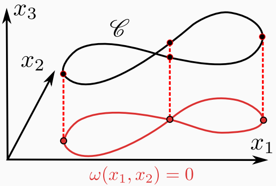


One-dimensional parametrization of $\mathscr{C} \subset \mathbb{C}^n$ algebraic curve

$(\omega, \rho_3, \ldots, \rho_n) \subset \mathbb{Z}[x_1, x_2]$ s.t.

$$\mathscr{C} = \overline{\left\{ \begin{array}{c} \left(\boldsymbol{x}_1, \boldsymbol{x}_2, \frac{\rho_3(\boldsymbol{x}_1,\boldsymbol{x}_2)}{\partial_{x_2}\omega(\boldsymbol{x}_1,\boldsymbol{x}_2)}, \ldots, \frac{\rho_n(\boldsymbol{x}_1,\boldsymbol{x}_2)}{\partial_{x_2}\omega(\boldsymbol{x}_1,\boldsymbol{x}_2)} \right) \\ \text{s.t. } \omega(\boldsymbol{x}_1, \boldsymbol{x}_2) = 0 \quad \text{and} \quad \partial_{x_2}\omega(\boldsymbol{x}_1, \boldsymbol{x}_2) \neq 0 \end{array} \right\}}^{Z}$$



Data representation and quantitative estimate

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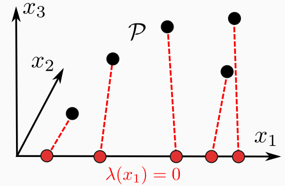

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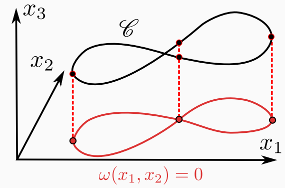


Magnitude of a polynomial

$\boldsymbol{f} \in \mathbb{Z}[x_1, \ldots, x_n]$ has *magnitude* (δ, τ) if

$\deg(\boldsymbol{f}) \leq \delta \quad$ and $\quad |\text{coeffs}(\boldsymbol{f})| \leq 2^{\tau}$

Soft-O notation

$$\tilde{O}(N) = O(N \log(N)^a)$$

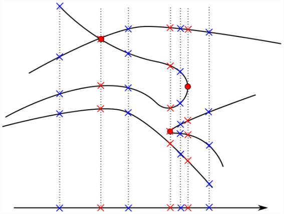
Results

Data

- $\mathscr{R} \subset \mathbb{Z}[x_1, x_2]$ of magnitude (δ, τ), encoding an algebraic curve $\mathscr{C} \subset \mathbb{C}^n$;
- $\mathscr{P} \subset \mathbb{Z}[x_1]$ of magnitude (δ, τ), encoding a finite $\mathcal{P} \subset \mathscr{C}$;

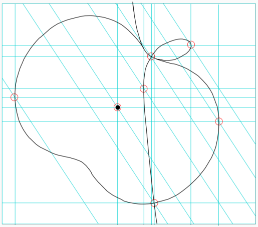
Computing topology

Ambient dimension	Bit complexity	Reference
$n = 2$	$\tilde{O}(\delta^5(\delta + \tau))$	[Kobel, Sagraloff; '15] [D.Diatta, S.Diatta, Rouiller, Roy, Sagraloff; '22]



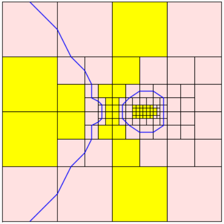
Cylindrical Algebraic Decomposition

[Collins, '75] [Kerber, Sagraloff; '12]



Multiple projections

[Seidel, Wolpert; '05]



Subdivision

[Burr, Choi, Galehouse, Yap; '05]

Results

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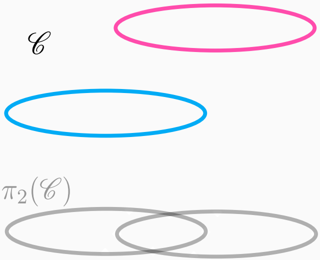

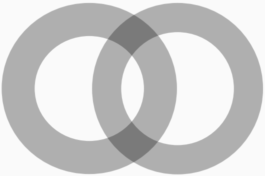

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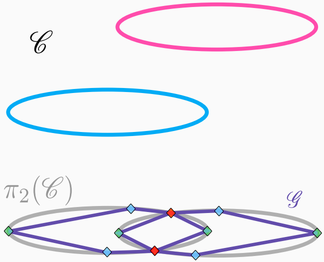

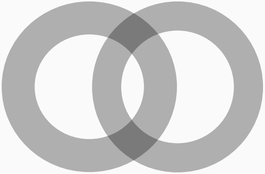

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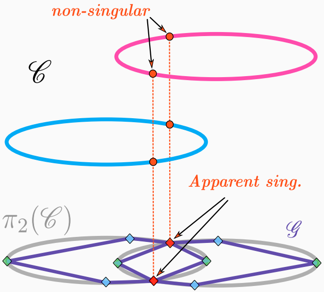

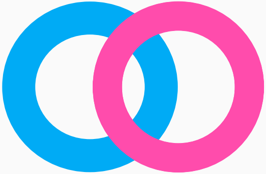

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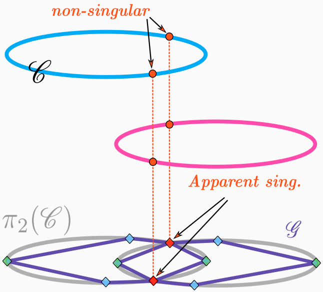

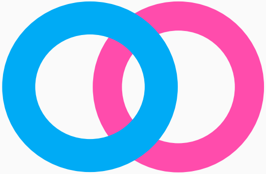

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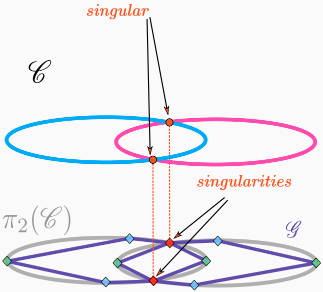

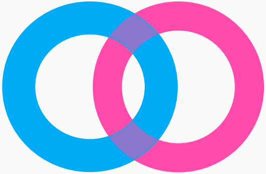

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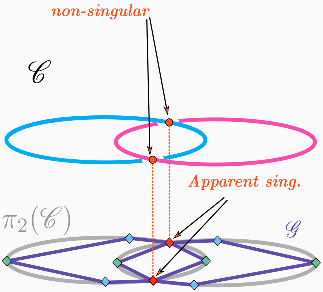

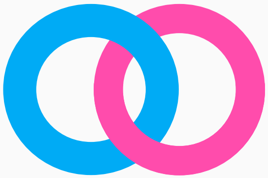

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Computing connectivity - Main Result NEW!

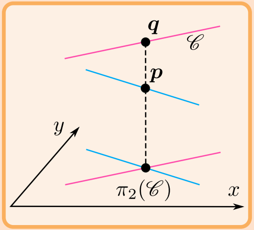
Ambient dimension	Bit complexity	Reference
$n \geq 2$	$\tilde{O}(\delta^5(\delta + \tau))$	[Islam, Poteaux, P.; 2023]

Avoid computation of the complete topology!

Apparent singularities: key idea

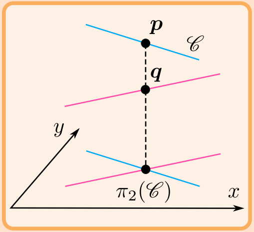
Generic apparent singularities NEW!

Projecting in a generic direction introduce finitely many apparent singularities like:



Below

$\neq$

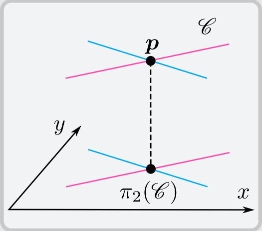


Above

$\neq$

Space singularities

Spatial nodes project as:



Same

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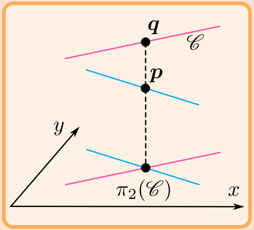
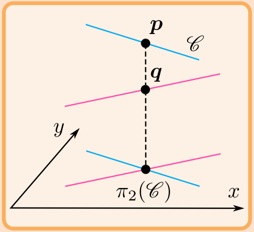
Generic apparent singularities NEW!

Projecting in a generic direction introduce finitely many apparent singularities like:



$\neq$

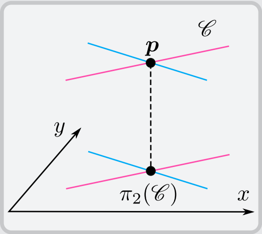


Apparent singularity   Apparent singularity

$\neq$

Space singularities

Spatial nodes project as:



Actual singularity

Key idea

Local connectivity does not depend on the relative position

Only two cases to consider!

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- $\mathscr{R} \subset \mathbb{Z}[x_1, x_2]$ of magnitude (δ, τ), encoding an algebraic curve $\mathscr{C} \subset \mathbb{C}^n$;
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- $\mathscr{C}$ satisfies genericity assumptions w.r.t. $\mathcal{P}$

Output

A partition of $\mathcal{P} \cap \mathbb{R}^n$ w.r.t. the connected components of $\mathscr{C} \cap \mathbb{R}^n$.

1. $\mathscr{D}, \mathscr{Q} \leftarrow$ Proj2D($\mathscr{R}$), Proj2D($\mathscr{P}$);
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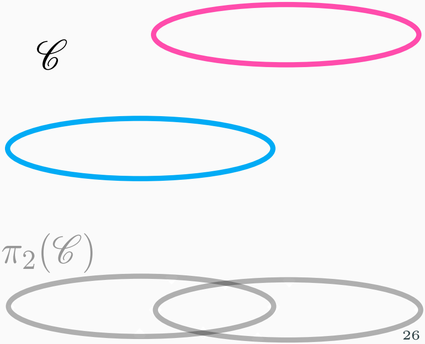

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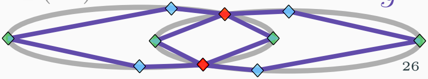
$\mathscr{C}$

- $\rightsquigarrow$ resultants
- $\rightsquigarrow$ $\mathbb{R}$-root isolation
- univariate
- bivariate triangular

Planar topology

$\tilde{O}(\delta^5(\delta + \tau))$



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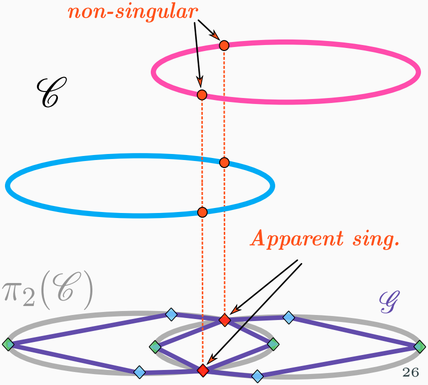

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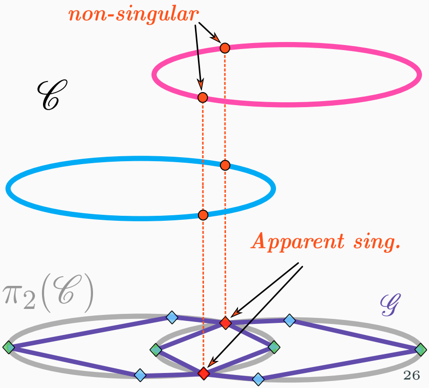

⇝ subresultant seqs
⇝ GCD computations
⇝ multi-modularization

Apparent sing. identification

$\tilde{O}(\delta^5(\delta + \tau))$

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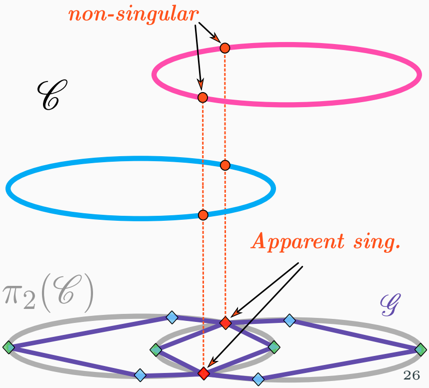

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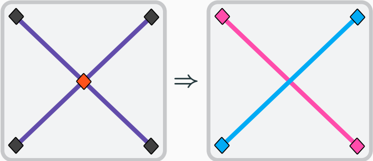

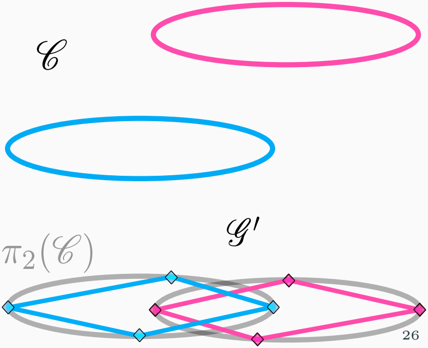

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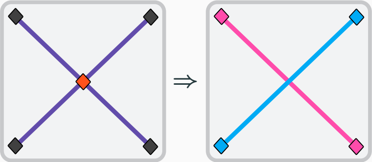

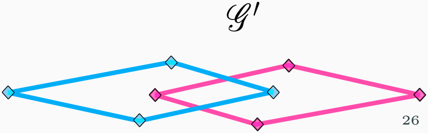

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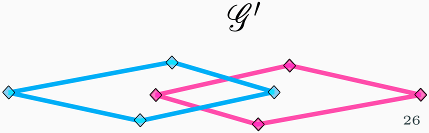
Overall Complexity

$\tilde{O}(\delta^5(\delta + \tau))$



Summary

Input

Polynomials in $\mathbb{Q}[x_1, \ldots x_n]$ of max degree D defining a smooth algebraic set of dim. d

Connectivity reduction process - before

Arbitrary dimension $\overset{\text{ROADMAP}}{\Longrightarrow}$ Dimension: 1, Size: $(nD)^{O(n\log(n))}$ $\overset{\text{Topology}}{\Longrightarrow}$ Finite graph $\mathscr{G}$

ROADMAP ↓ $(nD)^{O(n\log^2(n))}$ [Basu, Roy; 2014]

Topology ↓ $(\text{Size})^{O(1)} = (nD)^{O(n\log(n))}$ [Safey El Din, Schost; 2011]

Connectivity reduction process - now

Arbitrary dimension $\overset{\text{ROADMAP}}{\Longrightarrow}$ Dimension: 1, Size: $(n^2D)^{4n\log_2(d)+O(n)}$ $\overset{\text{Topology}}{\Longrightarrow}$ Finite graph $\mathscr{G}$

ROADMAP ↓ $(n^2D)^{6n\log_2(d)+O(n)}$ [P., Safey El Din, Schost; 2024]

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Connectivity reduction process - now

Arbitrary dimension $\overset{\text{ROADMAP}}{\Longrightarrow}$ Dimension: 1, Size: $(n^2D)^{4n\log_2(d)+O(n)}$ $\overset{\text{Connectivity}}{\Longrightarrow}$ Finite graph $\mathscr{G}$

ROADMAP ↓ $(n^2D)^{6n\log_2(d)+O(n)}$ [P., Safey El Din, Schost; 2024]

Connectivity ↓ $(\text{Size})^3 = (n^2D)^{12n\log_2(d)+O(n)}$ [Islam, Poteaux, P.; 2023]

Algorithm for connectivity queries on real algebraic curves, 2023
with Md N. Islam and A. Poteaux

Contributions

Robotics applications

- ✓ First **cuspidality** decision algorithm with singly exponential bit-complexity
- ✓ Roadmap **computation** for a challenging robotics problem

Computational real algebraic geometry can solve actual problems in robotics

Improve connectivity queries solving

- ✓ Nearly optimal **roadmap** algorithm for unbounded algebraic sets
  $\rightsquigarrow$ Complexity: $(n^2D)^{6n\log_2 d+O(n)}$ $\rightsquigarrow$ Output size: $(n^2D)^{4n\log_2 d+O(n)}$
- ✓ Efficient algorithm for connectivity of real algebraic **curves**
  $\rightsquigarrow$ Complexity: $\tilde{O}(\delta^6)$

We have efficient algorithms for analyzing connectivity of real algebraic sets

Perspectives

Algorithms

Roadmap algorithms:

- Adapt the algorithms to structured systems: quadratic case (J.A.K.Elliott, M.Safey El Din, É.Schost)
- Reduce the size of the roadmap by taking fewer fibers (M.Safey El Din, É.Schost)
- Generalize the connectivity result to semi-algebraic sets
- Design optimal roadmap algorithms with complexity exponential in $O(n)$

Connectivity of s.a. curves:

- Obtain a deterministic version of the algorithm (F.Bréhard, A.Poteaux)
- Adapt to algebraic curves given as union (A.Poteaux)
- Generalize to semi-algebraic curves
- Investigate the connectivity of plane curves

Applications

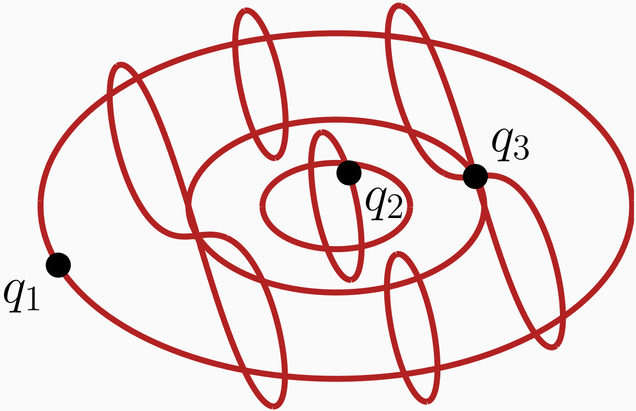
- Analyze challenging class of robots (D.Salunkhe, P.Wenger)
- Algorithms for rigidity and program verification problems (E.Bayarmagnai, F.Mohammadi)
- Obtain practical version of modern roadmap algorithms

Software

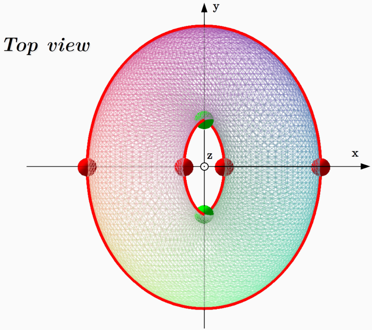
- Connectivity of curves: subresultant/GCD computations deg ~ 100 (now) $\rightarrow \sim 200$ (target)
- Build a Julia library for computational real algebraic geometry (C.Eder, R.Mohr)
- Implement a ready-to-use toolbox for roboticians

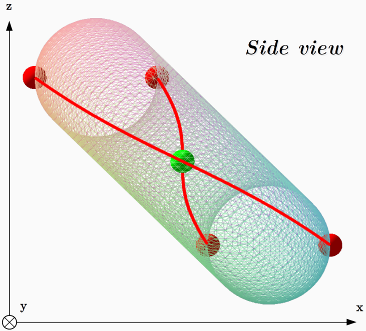
Union of curves

- Expected additional cost: compute all intersection points between curves, including these points as control points.



Reduce data size





Structured systems

$\deg(W(\pi_1, V)) \leq \binom{n-1}{p-1} D^p (D-1)^{n-p}$
If $D = 2$ then, the bound becomes $\binom{n-1}{p-1} 2^p$
We expect then a complexity $(nD)^{p \log_2(n-p)}$ for computing roadmaps

Toward roadmap algorithms for s.a. sets

Semi-algebraic sets

A strategy to tackle unbounded semi-algebraic sets:

$f \in \mathbb{R}[x_1, \ldots, x_n]$	$f \neq 0 \longrightarrow f \cdot u - 1 = 0$
u new variable	$f \geq 0 \longrightarrow f - u^2 = 0$
	$f > 0 \longrightarrow f \cdot u^2 - 1 = 0$

Thom's isotopy lemma

Set of proper points prop$(\mathcal{R}, V)$

$\boldsymbol{y}$ proper point of $\mathcal{R}_{|V}$ if there exists a ball $B \ni \boldsymbol{y}$

s.t. $\mathcal{R}^{-1}(B) \cap V$ *is closed and bounded*

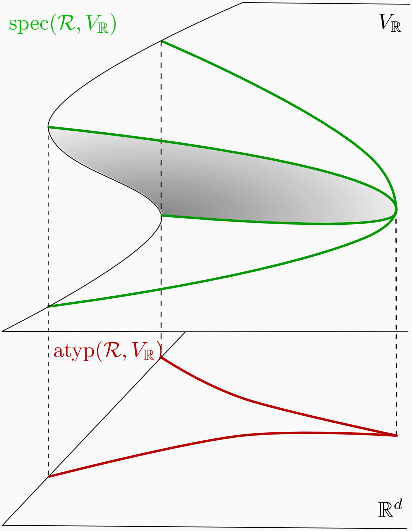
Atypical Values

$$\mathrm{atyp}(\mathcal{R}, V) = \mathrm{sval}(\mathcal{R}, V) \cup \left[\mathbb{C}^d - \mathrm{prop}(\mathcal{R}, V)\right]$$

Special Points

$$\mathrm{spec}(\mathcal{R}, V) = \mathcal{R}^{-1}\big(\mathrm{atyp}(\mathcal{R}, V)\big) \cap V$$

Semi-algebraic Thom's isotopy lemma [Coste & Shiota, 1995]



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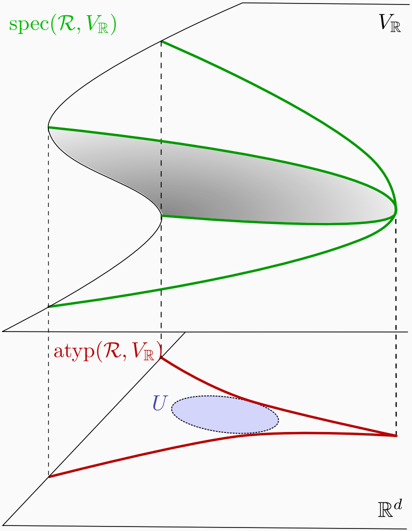

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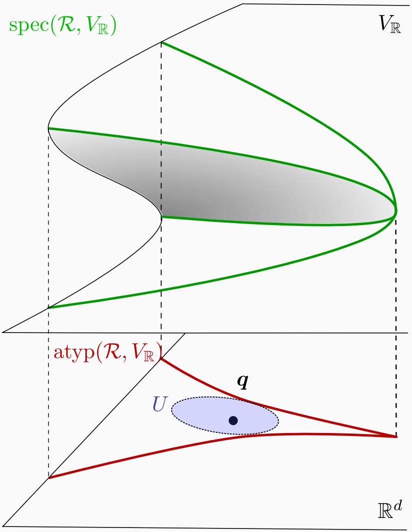

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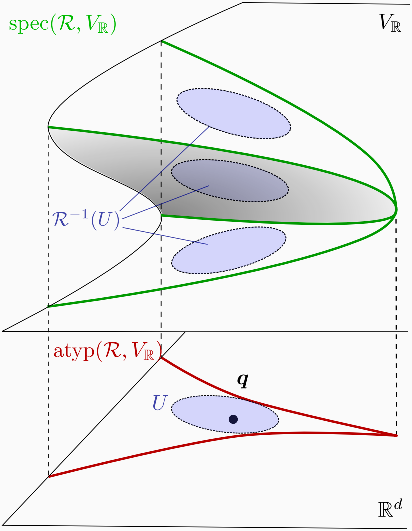

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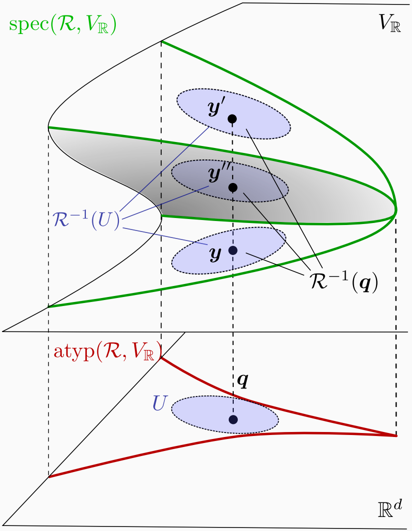

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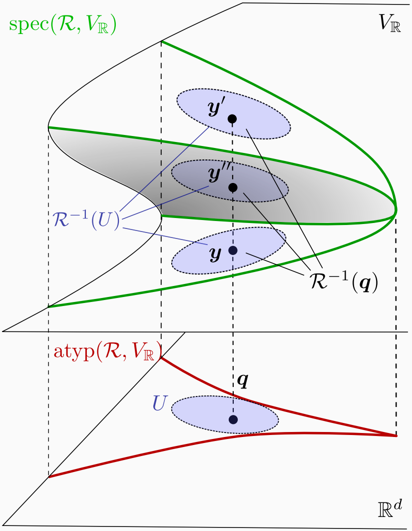
For any open connected subset $U \subset \mathbb{R}^d$ s.t $U \cap \text{atyp}(\mathcal{R}, V) = \emptyset$ and for any $\boldsymbol{q} \in U$, there exists a homeomorphism

$$\Psi : \left[\mathcal{R}^{-1}(U) \cap V_{\mathbb{R}}\right] \rightarrow \left[\mathcal{R}^{-1}(\boldsymbol{q}) \cap V_{\mathbb{R}}\right] \times U$$

such that the following diagram commutes

$$\begin{array}{ccc} \left[\mathcal{R}^{-1}(U) \cap V_{\mathbb{R}}\right] & \xrightarrow{\Psi} & \left[\mathcal{R}^{-1}(\boldsymbol{q}) \cap V_{\mathbb{R}}\right] \times U \\ & \searrow^{\mathcal{R}} & \downarrow \pi_U \\ & & U \end{array}$$



Cuspidality graph

Cuspidality graph

$\mathscr{G} = (\mathcal{P}, \mathcal{E})$ is a *cuspidality graph* of the restriction of $\mathcal{R}$ to $V_{\mathbb{R}}$ if the following holds

1. $\mathcal{P}$ intersects every connected component of $V_{\mathbb{R}} - \operatorname{spec}(\mathcal{R}, V)$
2. Let $\boldsymbol{p} \in \mathcal{P}$, then
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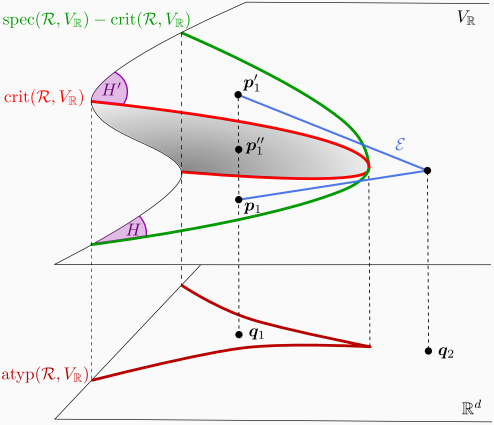

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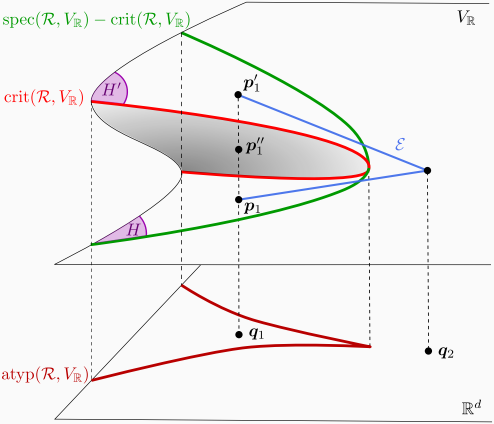

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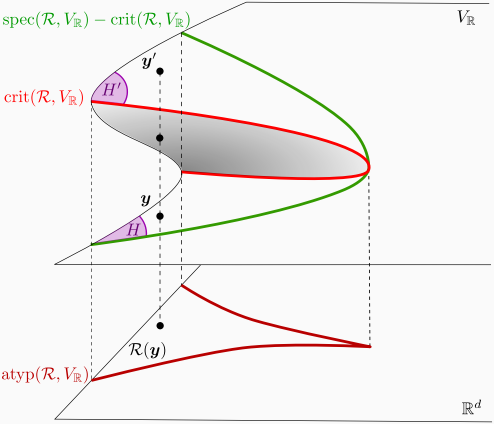

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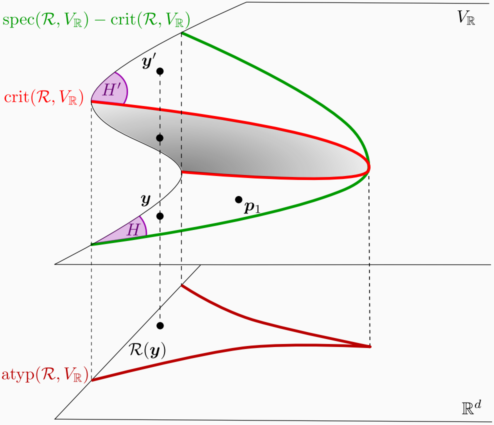

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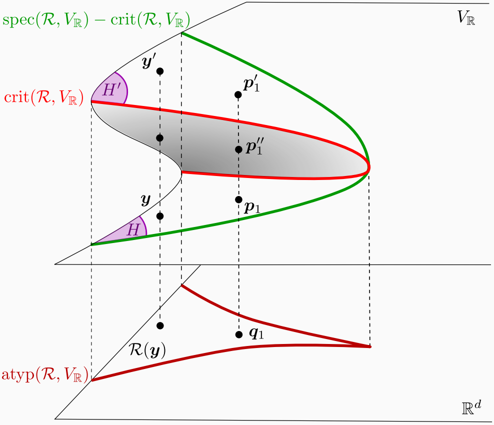

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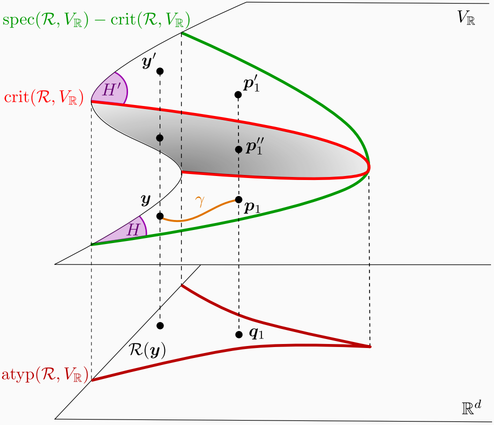

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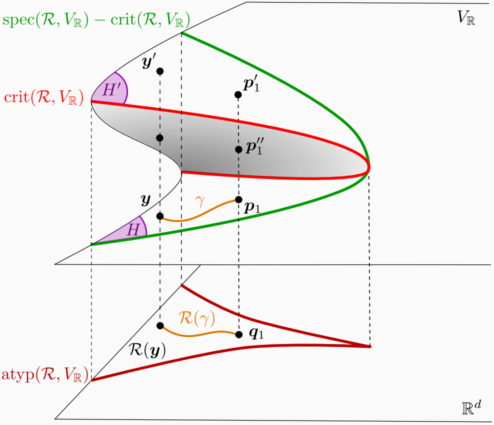

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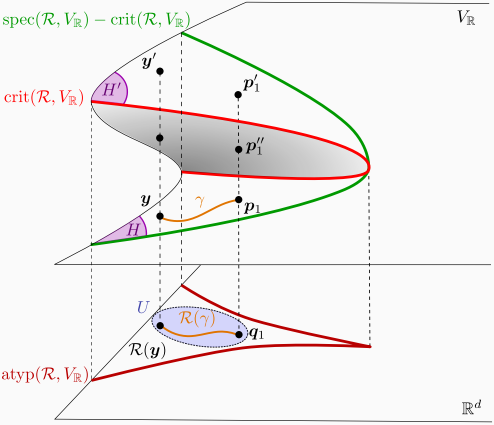

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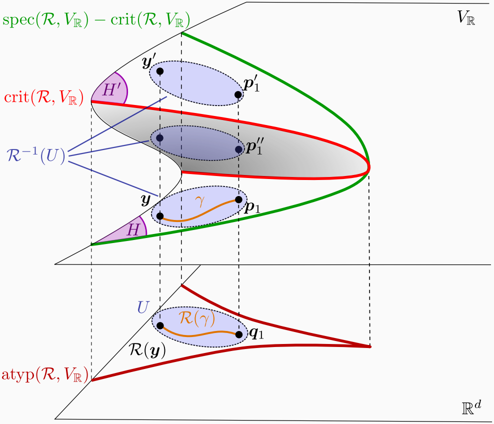

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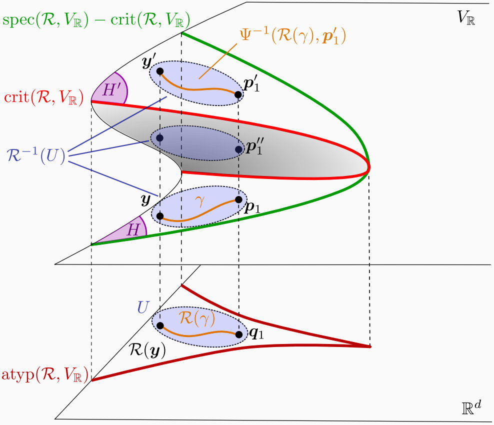

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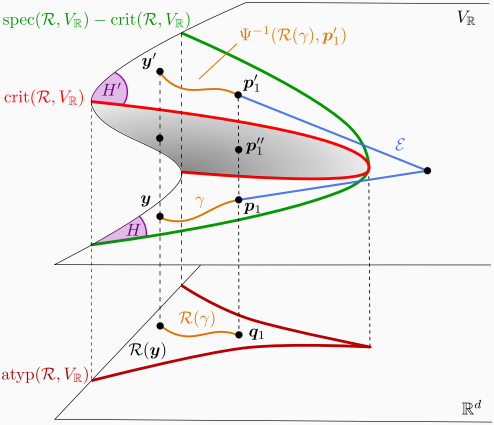

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Sample points algorithms

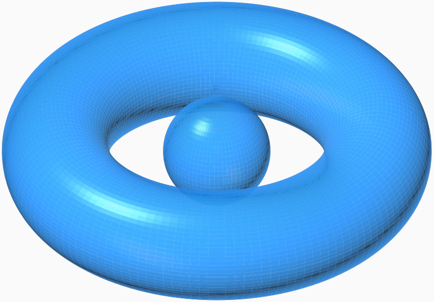
Semi-algebraic sets

$S \subset \mathbb{R}^{d}$ semi-algebraic set

$\Updownarrow$

Solution set of a finite system of polynomial equations $\boldsymbol{g}$ and inequalities $\boldsymbol{h}$

S has a finite number of connected components



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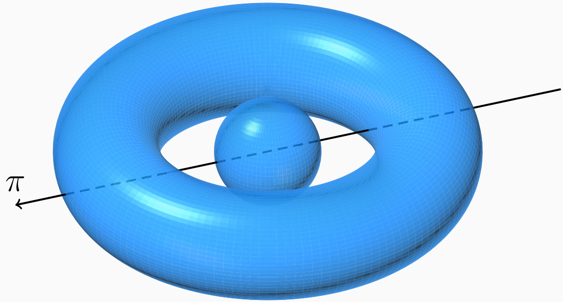

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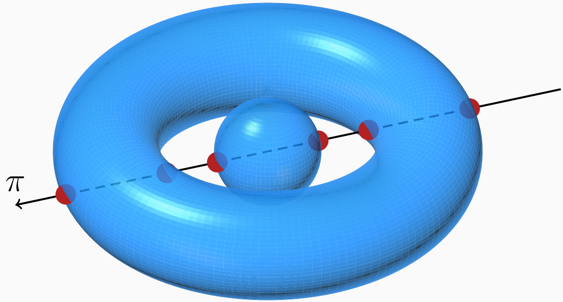

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Theorem [Basu & Pollack & Roy, 2016] [Le & Safey El Din, 2022]

- $S \subset \mathbb{R}^{\boldsymbol{d}}$ defined by $g_1 = \cdots = g_s = 0 \quad \text{and} \quad h_1 > 0, \ldots, h_t > 0$
- $D = \max(\deg(\boldsymbol{g}), \deg(\boldsymbol{h}))$
- $\tau = \max\{\text{bitsize of the input coefficients}\}$

There exists an algorithm SamplePoints s.t. if $\mathcal{Q} \leftarrow \textsc{SamplePoints}(\boldsymbol{f}, \boldsymbol{g})$ then

1. $\mathcal{Q} \subset S$ is finite
2. $\mathcal{Q}$ meets every connected component of S
3. $\text{card}(\mathcal{Q}) \leq D^{O(\boldsymbol{d})}$

Bit complexity of SamplePoints: $\tau(tD)^{O(\boldsymbol{d})}$

The cuspidality decision algorithm

Input

- $\boldsymbol{f} = (f_1, \ldots, f_s)$ and $\mathcal{R} = (r_1, \ldots, r_d)$ polynomials in $\mathbb{R}[x_1, \ldots, x_n]$
- $V = \boldsymbol{V}(\boldsymbol{f})$ and $V_{\mathbb{R}} = V \cap \mathbb{R}^n$ are equidimensional of dimension d
- $D = \max\{\deg \boldsymbol{f}, \deg \mathcal{R}\}$ $\quad \tau = \max\{\text{bitsize of the input coefficients}\}$

Output

A decision, `True` or `False`, on the cuspidality of the restriction of $\mathcal{R}$ to $V_{\mathbb{R}}$.

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2. $\mathcal{Q} \leftarrow$ SamplePoints($\pm\boldsymbol{g}$); [Basu & Pollack & Roy, '16]$\Rightarrow \tau n^{O(d^2)} D^{O(nd)}$

[Le & Safey El Din, '21][Jelonek & Kurdyka, '05] $\nearrow$

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4. $\Delta \leftarrow$ Crit$(\mathcal{R}, \boldsymbol{f})$;

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1. $\boldsymbol{g} \leftarrow$ AtypicalValues$(\mathcal{R}, \boldsymbol{f})$; [Basu & Pollack & Roy, '16]$\Rightarrow \tau(sD)^{O(nd)}$
2. $\mathcal{Q} \leftarrow$ SamplePoints$(\pm\boldsymbol{g})$; [Basu & Pollack & Roy, '16]$\Rightarrow \tau n^{O(d^2)} D^{O(nd)}$
3. $\mathcal{P} \leftarrow \mathcal{R}^{-1}(\mathcal{Q})$; [Le & Safey El Din, '21][Jelonek & Kurdyka, '05] $\nearrow\!\!\!\nearrow$
4. $\Delta \leftarrow$ Crit$(\mathcal{R}, \boldsymbol{f})$;
5. $\mathscr{R} \leftarrow$ Roadmap$(\boldsymbol{f}, \pm\Delta, \mathcal{P})$; [Basu & Pollack & Roy, '00]$\Rightarrow \tilde{O}(\tau) \cdot ((s+d)D)^{O(n^2)}$

The cuspidality decision algorithm

Input

- $\boldsymbol{f} = (f_1, \dots, f_s)$ and $\mathcal{R} = (r_1, \dots, r_d)$ polynomials in $\mathbb{R}[x_1, \dots, x_n]$
- $V = \boldsymbol{V}(\boldsymbol{f})$ and $V_{\mathbb{R}} = V \cap \mathbb{R}^n$ are equidimensional of dimension d
- $D = \max\{\deg \boldsymbol{f}, \deg \mathcal{R}\}$ $\quad \tau = \max\{\text{bitsize of the input coefficients}\}$

Output

A decision, **`True`** or **`False`**, on the cuspidality of the restriction of $\mathcal{R}$ to $V_{\mathbb{R}}$.

1. $\boldsymbol{g} \leftarrow$ AtypicalValues$(\mathcal{R}, \boldsymbol{f})$; [Basu & Pollack & Roy, '16]$\Rightarrow \tau(sD)^{O(nd)}$
2. $\mathcal{Q} \leftarrow$ SamplePoints$(\pm\boldsymbol{g})$; [Basu & Pollack & Roy, '16]$\Rightarrow \tau n^{O(d^2)} D^{O(nd)}$
3. $\mathcal{P} \leftarrow \mathcal{R}^{-1}(\mathcal{Q})$; [Le & Safey El Din, '21][Jelonek & Kurdyka, '05] $\nearrow$
4. $\Delta \leftarrow$ Crit$(\mathcal{R}, \boldsymbol{f})$;
5. $\mathscr{R} \leftarrow$ Roadmap$(\boldsymbol{f}, \pm\Delta, \mathcal{P})$; [Basu & Pollack & Roy, '00]$\Rightarrow \tilde{O}(\tau) \cdot ((s+d)D)^{O(n^2)}$
6. $\mathscr{G} = (\mathcal{P}, \mathcal{E}) \leftarrow$ GraphIsotop$(\mathscr{R}, \pm\Delta, \mathcal{P})$;

The cuspidality decision algorithm

Input

- $\boldsymbol{f} = (f_1, \ldots, f_s)$ and $\mathcal{R} = (r_1, \ldots, r_d)$ polynomials in $\mathbb{R}[x_1, \ldots, x_n]$
- $V = \boldsymbol{V}(\boldsymbol{f})$ and $V_{\mathbb{R}} = V \cap \mathbb{R}^n$ are equidimensional of dimension d
- $D = \max\{\deg \boldsymbol{f}, \deg \mathcal{R}\}$ $\quad \tau = \max\{\text{bitsize of the input coefficients}\}$

Output

A decision, `True` or `False`, on the cuspidality of the restriction of $\mathcal{R}$ to $V_{\mathbb{R}}$.

1. $\boldsymbol{g} \leftarrow$ AtypicalValues$(\mathcal{R}, \boldsymbol{f})$; [Basu & Pollack & Roy, '16]$\Rightarrow \tau(sD)^{O(nd)}$
2. $\mathcal{Q} \leftarrow$ SamplePoints$(\pm \boldsymbol{g})$; [Basu & Pollack & Roy, '16]$\Rightarrow \tau n^{O(d^2)} D^{O(nd)}$
3. $\mathcal{P} \leftarrow \mathcal{R}^{-1}(\mathcal{Q})$; [Le & Safey El Din, '21][Jelonek & Kurdyka, '05] $\nearrow$
4. $\Delta \leftarrow$ Crit$(\mathcal{R}, \boldsymbol{f})$;
5. $\mathscr{R} \leftarrow$ Roadmap$(\boldsymbol{f}, \pm\Delta, \mathcal{P})$; [Basu & Pollack & Roy, '00]$\Rightarrow \tilde{O}(\tau) \cdot ((s+d)D)^{O(n^2)}$
6. $\mathscr{G} = (\mathcal{P}, \mathcal{E}) \leftarrow$ GraphIsotop$(\mathscr{R}, \pm\Delta, \mathcal{P})$;
7. for $\boldsymbol{q} \in \mathcal{Q}$ do

The cuspidality decision algorithm

Input

- $\boldsymbol{f} = (f_1, \dots, f_s)$ and $\mathcal{R} = (r_1, \dots, r_d)$ polynomials in $\mathbb{R}[x_1, \dots, x_n]$
- $V = \boldsymbol{V}(\boldsymbol{f})$ and $V_{\mathbb{R}} = V \cap \mathbb{R}^n$ are equidimensional of dimension d
- $D = \max\{\deg \boldsymbol{f}, \deg \mathcal{R}\}$ $\quad \tau = \max\{\text{bitsize of the input coefficients}\}$

Output

A decision, `True` or `False`, on the cuspidality of the restriction of $\mathcal{R}$ to $V_{\mathbb{R}}$.

1. $\boldsymbol{g} \leftarrow$ AtypicalValues($\mathcal{R}, \boldsymbol{f}$); [Basu & Pollack & Roy, '16]$\Rightarrow \tau(sD)^{O(nd)}$
2. $\mathcal{Q} \leftarrow$ SamplePoints($\pm \boldsymbol{g}$); [Basu & Pollack & Roy, '16]$\Rightarrow \tau n^{O(d^2)} D^{O(nd)}$
3. $\mathcal{P} \leftarrow \mathcal{R}^{-1}(\mathcal{Q})$; [Le & Safey El Din, '21][Jelonek & Kurdyka, '05] $\nearrow$
4. $\Delta \leftarrow$ Crit($\mathcal{R}, \boldsymbol{f}$);
5. $\mathscr{R} \leftarrow$ Roadmap($\boldsymbol{f}, \pm\Delta, \mathcal{P}$); [Basu & Pollack & Roy, '00]$\Rightarrow \tilde{O}(\tau) \cdot ((s+d)D)^{O(n^2)}$
6. $\mathscr{G} = (\mathcal{P}, \mathcal{E}) \leftarrow$ GraphIsotop($\mathscr{R}, \pm\Delta, \mathcal{P}$);
7. for $\boldsymbol{q} \in \mathcal{Q}$ do
8. $\quad$ for $\boldsymbol{v}_1 \neq \boldsymbol{v}_2 \in \mathcal{P} \cap \mathcal{R}^{-1}(\boldsymbol{q})$ do

The cuspidality decision algorithm

Input

- $\boldsymbol{f} = (f_1, \ldots, f_s)$ and $\mathcal{R} = (r_1, \ldots, r_d)$ polynomials in $\mathbb{R}[x_1, \ldots, x_n]$
- $V = \boldsymbol{V}(\boldsymbol{f})$ and $V_{\mathbb{R}} = V \cap \mathbb{R}^n$ are equidimensional of dimension d
- $D = \max\{\deg \boldsymbol{f}, \deg \mathcal{R}\}$ $\quad \tau = \max\{\text{bitsize of the input coefficients}\}$

Output

A decision, `True` or `False`, on the cuspidality of the restriction of $\mathcal{R}$ to $V_{\mathbb{R}}$.

1. $\boldsymbol{g} \leftarrow$ AtypicalValues$(\mathcal{R}, \boldsymbol{f})$; [Basu & Pollack & Roy, '16]$\Rightarrow \tau(sD)^{O(nd)}$
2. $\mathcal{Q} \leftarrow$ SamplePoints$(\pm \boldsymbol{g})$; [Basu & Pollack & Roy, '16]$\Rightarrow \tau n^{O(d^2)} D^{O(nd)}$
3. $\mathcal{P} \leftarrow \mathcal{R}^{-1}(\mathcal{Q})$; [Le & Safey El Din, '21][Jelonek & Kurdyka, '05] $\nearrow$
4. $\Delta \leftarrow$ Crit$(\mathcal{R}, \boldsymbol{f})$;
5. $\mathscr{R} \leftarrow$ Roadmap$(\boldsymbol{f}, \pm\Delta, \mathcal{P})$; [Basu & Pollack & Roy, '00]$\Rightarrow \tilde{O}(\tau) \cdot ((s+d)D)^{O(n^2)}$
6. $\mathscr{G} = (\mathcal{P}, \mathcal{E}) \leftarrow$ GraphIsotop$(\mathscr{R}, \pm\Delta, \mathcal{P})$;
7. for $\boldsymbol{q} \in \mathcal{Q}$ do
8. for $\boldsymbol{v}_1 \neq \boldsymbol{v}_2 \in \mathcal{P} \cap \mathcal{R}^{-1}(\boldsymbol{q})$ do
9. if $\boldsymbol{v}_1, \boldsymbol{v}_2$ are connected in $\mathscr{G}$ then

The cuspidality decision algorithm

Input

- $\boldsymbol{f} = (f_1, \ldots, f_s)$ and $\mathcal{R} = (r_1, \ldots, r_d)$ polynomials in $\mathbb{R}[x_1, \ldots, x_n]$
- $V = \boldsymbol{V}(\boldsymbol{f})$ and $V_{\mathbb{R}} = V \cap \mathbb{R}^n$ are equidimensional of dimension d
- $D = \max\{\deg \boldsymbol{f}, \deg \mathcal{R}\}$ $\tau = \max\{\text{bitsize of the input coefficients}\}$

Output

A decision, `True` or `False`, on the cuspidality of the restriction of $\mathcal{R}$ to $V_{\mathbb{R}}$.

1. $\boldsymbol{g} \leftarrow$ AtypicalValues($\mathcal{R}, \boldsymbol{f}$); [Basu & Pollack & Roy, '16]$\Rightarrow \tau(sD)^{O(nd)}$
2. $\mathcal{Q} \leftarrow$ SamplePoints($\pm\boldsymbol{g}$); [Basu & Pollack & Roy, '16]$\Rightarrow \tau n^{O(d^2)} D^{O(nd)}$
3. $\mathcal{P} \leftarrow \mathcal{R}^{-1}(\mathcal{Q})$; [Le & Safey El Din, '21][Jelonek & Kurdyka, '05] $\nearrow$
4. $\Delta \leftarrow$ Crit($\mathcal{R}, \boldsymbol{f}$);
5. $\mathscr{R} \leftarrow$ Roadmap($\boldsymbol{f}, \pm\Delta, \mathcal{P}$); [Basu & Pollack & Roy, '00]$\Rightarrow \tilde{O}(\tau) \cdot ((s+d)D)^{O(n^2)}$
6. $\mathscr{G} = (\mathcal{P}, \mathcal{E}) \leftarrow$ GraphIsotop($\mathscr{R}, \pm\Delta, \mathcal{P}$);
7. for $\boldsymbol{q} \in \mathcal{Q}$ do
8.   for $\boldsymbol{v}_1 \neq \boldsymbol{v}_2 \in \mathcal{P} \cap \mathcal{R}^{-1}(\boldsymbol{q})$ do
9.   if $\boldsymbol{v}_1, \boldsymbol{v}_2$ are connected in $\mathscr{G}$ then
10.   return `True`;

The cuspidality decision algorithm

Input

- $\boldsymbol{f} = (f_1, \ldots, f_s)$ and $\mathcal{R} = (r_1, \ldots, r_d)$ polynomials in $\mathbb{R}[x_1, \ldots, x_n]$
- $V = \boldsymbol{V}(\boldsymbol{f})$ and $V_\mathbb{R} = V \cap \mathbb{R}^n$ are equidimensional of dimension d
- $D = \max\{\deg \boldsymbol{f}, \deg \mathcal{R}\}$ $\quad \tau = \max\{\text{bitsize of the input coefficients}\}$

Output

A decision, `True` or `False`, on the cuspidality of the restriction of $\mathcal{R}$ to $V_\mathbb{R}$.

1. $\boldsymbol{g} \leftarrow$ ATYPICALVALUES$(\mathcal{R}, \boldsymbol{f})$; [Basu & Pollack & Roy, '16]$\Rightarrow \tau (sD)^{O(nd)}$
2. $\mathcal{Q} \leftarrow$ SAMPLEPOINTS$(\pm \boldsymbol{g})$; [Basu & Pollack & Roy, '16]$\Rightarrow \tau n^{O(d^2)} D^{O(nd)}$
3. $\mathcal{P} \leftarrow \mathcal{R}^{-1}(\mathcal{Q})$; [Le & Safey El Din, '21][Jelonek & Kurdyka, '05] $\nearrow$
4. $\Delta \leftarrow$ CRIT$(\mathcal{R}, \boldsymbol{f})$;
5. $\mathscr{R} \leftarrow$ ROADMAP$(\boldsymbol{f}, \pm\Delta, \mathcal{P})$; [Basu & Pollack & Roy, '00]$\Rightarrow \tilde{O}(\tau) \cdot ((s+d)D)^{O(n^2)}$
6. $\mathscr{G} = (\mathcal{P}, \mathcal{E}) \leftarrow$ GRAPHISOTOP$(\mathscr{R}, \pm\Delta, \mathcal{P})$;
7. for $\boldsymbol{q} \in \mathcal{Q}$ do
8.   for $\boldsymbol{v}_1 \neq \boldsymbol{v}_2 \in \mathcal{P} \cap \mathcal{R}^{-1}(\boldsymbol{q})$ do
9.   if $\boldsymbol{v}_1, \boldsymbol{v}_2$ are connected in $\mathscr{G}$ then
10.   return `True`;
11. return `False`;

The cuspidality decision algorithm

Input

- $\boldsymbol{f} = (f_1, \ldots, f_s)$ and $\mathcal{R} = (r_1, \ldots, r_d)$ polynomials in $\mathbb{R}[x_1, \ldots, x_n]$
- $V = \boldsymbol{V}(\boldsymbol{f})$ and $V_{\mathbb{R}} = V \cap \mathbb{R}^n$ are equidimensional of dimension d
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$\mathcal{Q} \leftarrow$ SamplePoints($\pm \boldsymbol{g}$); [Basu & Pollack & Roy, '16]$\Rightarrow \tau n^{O(d^2)} D^{O(nd)}$

$\mathcal{P} \leftarrow \mathcal{R}^{-1}(\mathcal{Q})$; [Le & Safey El Din, '21][Jelonek & Kurdyka, '05] ⇗

$\Delta \leftarrow$ Crit($\mathcal{R}, \boldsymbol{f}$);

$\mathscr{R} \leftarrow$ Roadmap($\boldsymbol{f}, \pm\Delta, \mathcal{P}$); [Basu & Pollack & Roy, '00]$\Rightarrow \tilde{O}(\tau) \cdot ((s+d)D)^{O(n^2)}$

$\mathscr{G} = (\mathcal{P}, \mathcal{E}) \leftarrow$ GraphIsotop($\mathscr{R}, \pm\Delta, \mathcal{P}$);

for $\boldsymbol{q} \in \mathcal{Q}$ do

  for $\boldsymbol{v}_1 \neq \boldsymbol{v}_2 \in \mathcal{P} \cap \mathcal{R}^{-1}(\boldsymbol{q})$ do

    if $\boldsymbol{v}_1, \boldsymbol{v}_2$ are connected in $\mathscr{G}$ then

      return `True`;

return `False`;

Total bit complexity bound

$$\tilde{O}(\tau) \cdot ((s+d)D)^{O(n^2)}$$

The cuspidality decision algorithm

Input

- $\boldsymbol{f} = (f_1, \ldots, f_s)$ and $\mathcal{R} = (r_1, \ldots, r_d)$ polynomials in $\mathbb{R}[x_1, \ldots, x_n]$
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The cuspidality decision algorithm

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- $\boldsymbol{f} = (f_1, \ldots, f_s)$ and $\mathcal{R} = (r_1, \ldots, r_d)$ polynomials in $\mathbb{R}[x_1, \ldots, x_n]$
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for $\boldsymbol{q} \in \mathcal{Q}$ do

    for $\boldsymbol{v}_1 \neq \boldsymbol{v}_2 \in \mathcal{P} \cap \mathcal{R}^{-1}(\boldsymbol{q})$ do

        if $\boldsymbol{v}_1, \boldsymbol{v}_2$ are connected in $\mathscr{G}$ then

            return `True`;

return `False`;

Total bit complexity bound

$\tilde{O}(\tau) \cdot ((s+d)D)^{O(n^2)}$

Connectivity queries: algorithms

Data

- $S \subset \mathbb{R}^{\boldsymbol{n}}$ defined by $g_1 = \cdots = g_s = 0 \quad$ and $\quad h_1 > 0, \ldots, h_t > 0$
- $D = \max(\deg(\boldsymbol{g}), \deg(\boldsymbol{h}))$ and $\tau = \max\{\text{bitsize of the input coefficients}\}$
- $\mathcal{P} \subset V_{\mathbb{R}}$ of cardinality δ

$\oplus$

Connectivity queries: algorithms

Data

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- $\mathcal{P} \subset V_{\mathbb{R}}$ of cardinality δ

Theorem [Basu & Pollack & Roy, 2000]

There exists an algorithm RoadMap s.t if $\mathscr{R} \leftarrow \text{RoadMap}(\boldsymbol{g}, \boldsymbol{h}, \mathcal{P})$ then

1. $\mathscr{R} \subset S$ is a roadmap of $(S, \mathcal{P})$;
2. polynomials defining $\mathscr{R}$ have degrees $\leq t^{n+1} \delta D^{O(n^2)}$

Bit complexity of RoadMap:
$\leq \tilde{O}(\tau) \cdot t^{O(n)} \delta D^{O(n^2)}$

Connectivity queries: algorithms

Data

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- $D = \max(\deg(\boldsymbol{g}), \deg(\boldsymbol{h}))$ and $\tau = \max\{\text{bitsize of the input coefficients}\}$
- $\mathcal{P} \subset V_{\mathbb{R}}$ of cardinality δ

Theorem [Basu & Pollack & Roy, 2000]

There exists an algorithm ROADMAP s.t if $\mathscr{R} \leftarrow \text{ROADMAP}(\boldsymbol{g}, \boldsymbol{h}, \mathcal{P})$ then

1. $\mathscr{R} \subset S$ is a roadmap of $(S, \mathcal{P})$;
2. polynomials defining $\mathscr{R}$ have degrees $\leq t^{n+1}\delta D^{O(n^2)}$

Bit complexity of ROADMAP:
$\leq \tilde{O}(\tau) \cdot t^{O(n)}\delta D^{O(n^2)}$

$\oplus$

Theorem [Diatta & Mourrain & Ruatta, 2012] [Cheng & Jin & Lazard, 2013] [Jin & Cheng, 2021]

There exists an algorithm GRAPHISOTOP s.t if $\mathscr{G} \leftarrow \text{GRAPHISOTOP}(\mathscr{R}, \boldsymbol{h}, \mathcal{P})$ then

1. $\mathscr{G} = (\widetilde{\mathcal{P}}, \mathcal{E})$ is a graph s.t. $\mathcal{P} \subset \widetilde{\mathcal{P}}$
2. $\mathscr{G}$ is isotopy equivalent to $\mathscr{R} \cap S$

Bit complexity of GRAPHISOTOP:
$\leq \tilde{O}(\tau) \cdot (\delta \deg(\mathscr{R}))^{O(1)}$

Connectivity queries: algorithms

Data

- $S \subset \mathbb{R}^{n}$ defined by $g_1 = \cdots = g_s = 0$ and $h_1 > 0, \ldots, h_t > 0$
- $D = \max(\deg(\boldsymbol{g}), \deg(\boldsymbol{h}))$ and $\tau = \max\{\text{bitsize of the input coefficients}\}$
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Bit complexity of ROADMAP:
$\leq \tilde{O}(\tau) \cdot t^{O(n)}\delta D^{O(n^2)}$

$\oplus$

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Bit complexity of GRAPHISOTOP:
$\leq \tilde{O}(\tau) \cdot (\delta \deg(\mathscr{R}))^{O(1)}$

Connecting $\boldsymbol{p}, \boldsymbol{p}' \in \mathcal{P}$

$\boldsymbol{p}$ and $\boldsymbol{p}'$
path-connected in $S \iff$ path-connected in $\mathscr{R} \cap S \iff$ connected in $\mathscr{G}$

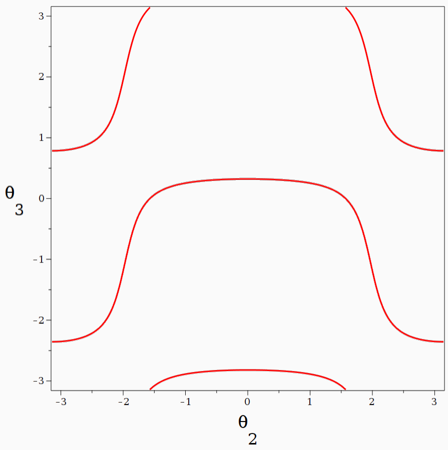
A basic cuspidal example

$$\mathcal{K}: \quad \begin{array}{ccc} \boldsymbol{R}^3 & \longrightarrow & \boldsymbol{R}^3 \\ \boldsymbol{\theta} & \longmapsto & \big(z_1(\boldsymbol{\theta}), z_2(\boldsymbol{\theta}), z_3(\boldsymbol{\theta})\big) \end{array}$$

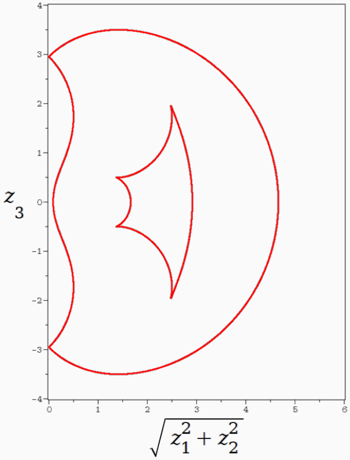
$$z_1 = \frac{1}{2}c_1 c_2(3c_3+4) - \frac{1}{2}s_1(3s_3+2) + c_1$$

$$z_2 = \frac{1}{2}s_1 c_2(3c_3+4) + \frac{1}{2}c_1(3s_3+2) + s_1$$

$$z_3 = -\frac{1}{2}s_2(3c_3+4)$$



$\xrightarrow{\mathcal{K}}$



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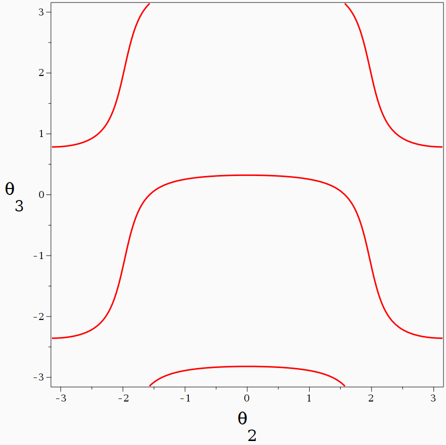
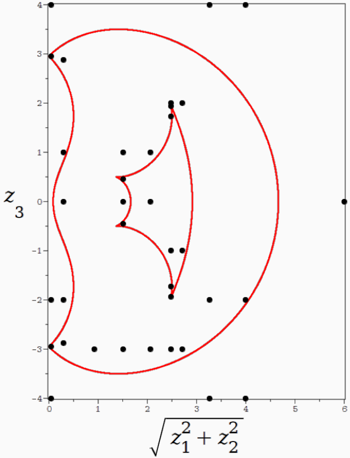


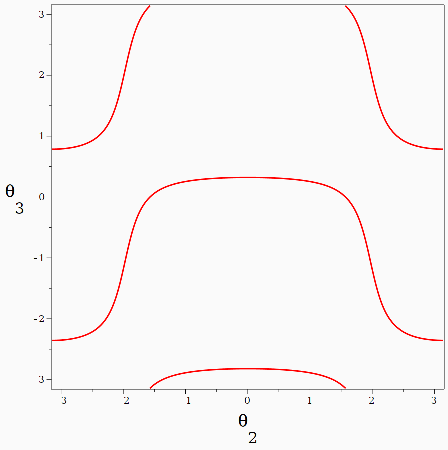
$\xrightarrow{\mathcal{K}}$



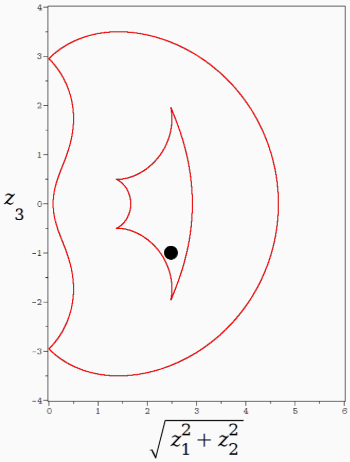
A basic cuspidal example

$$\mathcal{K}: \quad \boldsymbol{R}^3 \longrightarrow \boldsymbol{R}^3$$
$$\boldsymbol{\theta} \longmapsto \big(z_1(\boldsymbol{\theta}), z_2(\boldsymbol{\theta}), z_3(\boldsymbol{\theta})\big)$$

$$z_1 = \frac{1}{2}c_1c_2(3c_3+4) - \frac{1}{2}s_1(3s_3+2) + c_1$$
$$z_2 = \frac{1}{2}s_1c_2(3c_3+4) + \frac{1}{2}c_1(3s_3+2) + s_1$$
$$z_3 = -\frac{1}{2}s_2(3c_3+4)$$



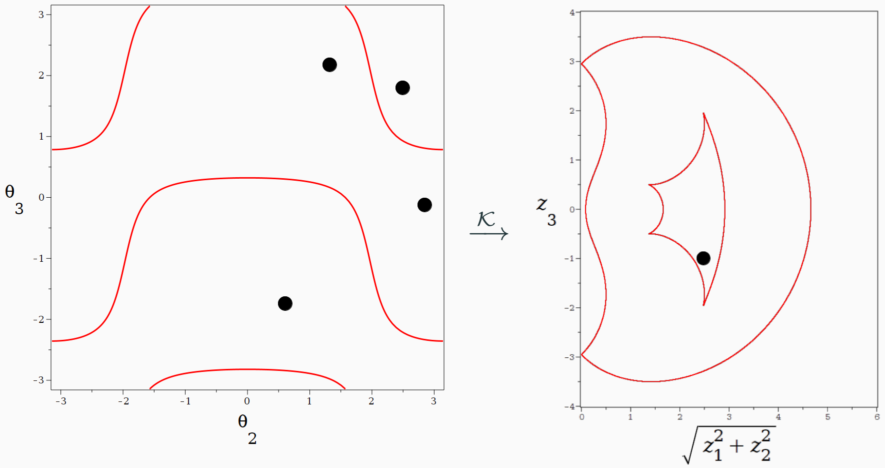
$\xrightarrow{\mathcal{K}}$



A basic cuspidal example

$$\mathcal{K}: \quad \begin{array}{ccc} \boldsymbol{R}^3 & \longrightarrow & \boldsymbol{R}^3 \\ \boldsymbol{\theta} & \longmapsto & \big(z_1(\boldsymbol{\theta}), z_2(\boldsymbol{\theta}), z_3(\boldsymbol{\theta})\big) \end{array}$$

$$\begin{aligned} z_1 &= \frac{1}{2} c_1 c_2 (3c_3 + 4) - \frac{1}{2} s_1 (3s_3 + 2) + c_1 \\ z_2 &= \frac{1}{2} s_1 c_2 (3c_3 + 4) + \frac{1}{2} c_1 (3s_3 + 2) + s_1 \\ z_3 &= -\frac{1}{2} s_2 (3c_3 + 4) \end{aligned}$$



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$$
\begin{array}{rccc}
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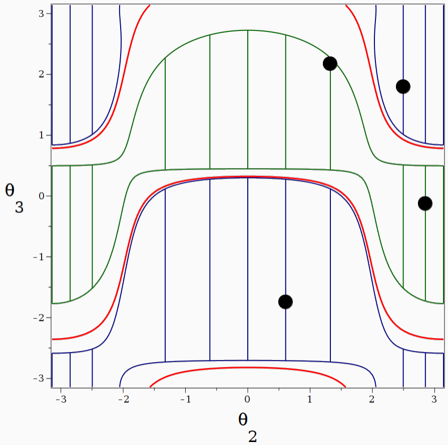
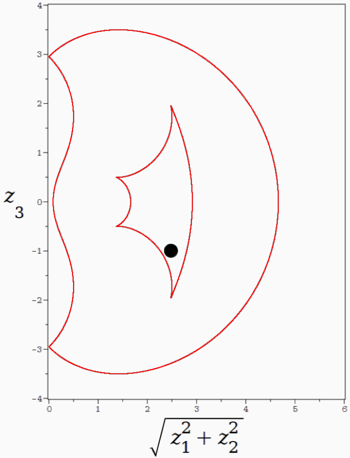


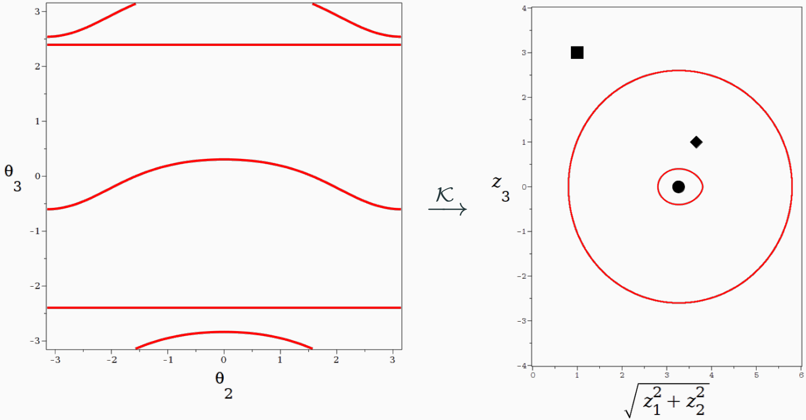
$\xrightarrow{\mathcal{K}}$



A basic non-cuspidal example

$$\mathcal{K}: \quad \begin{array}{ccc} \boldsymbol{R}^3 & \longrightarrow & \boldsymbol{R}^3 \\ \boldsymbol{\theta} & \longmapsto & \big(z_1(\boldsymbol{\theta}), z_2(\boldsymbol{\theta}), z_3(\boldsymbol{\theta})\big) \end{array}$$

$$\begin{aligned} z_1 &= \frac{1}{10} c_1 c_2 (15 c_3 + 11) - \frac{1}{10} s_1 (15 s_3 + 13) + 3 c_1 \\ z_2 &= \frac{1}{10} s_1 c_2 (15 c_3 + 11) + \frac{1}{10} c_1 (15 s_3 + 13) + 3 s_1 \\ z_3 &= -\frac{1}{10} s_2 (15 c_3 + 11) \end{aligned}$$



A basic non-cuspidal example

$$\mathcal{K}\colon \quad \boldsymbol{R}^3 \longrightarrow \boldsymbol{R}^3, \qquad \boldsymbol{\theta} \longmapsto \big(z_1(\boldsymbol{\theta}), z_2(\boldsymbol{\theta}), z_3(\boldsymbol{\theta})\big)$$

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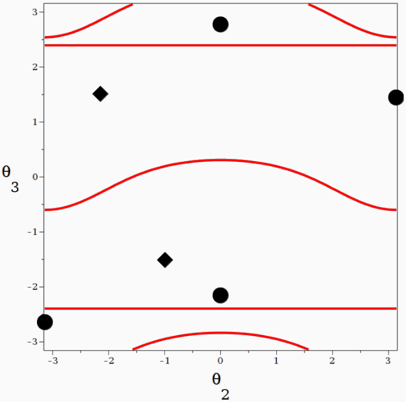

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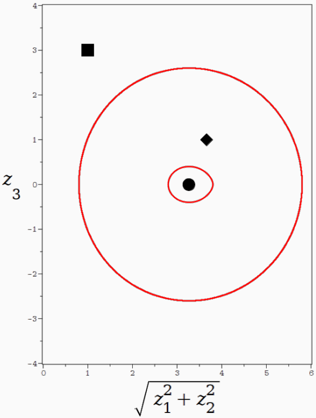

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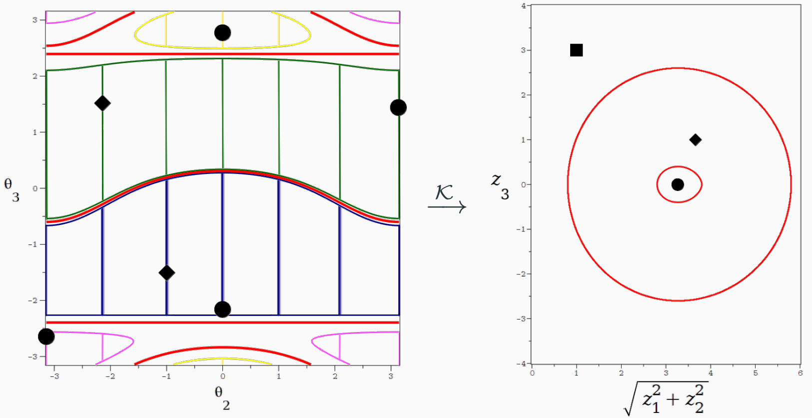

Proof of the new connectivity result

Non-negative proper polynomial map

$$\boldsymbol{\varphi}_i : \begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^i \\ \boldsymbol{x} & \mapsto & (\psi_1(\boldsymbol{x}), \ldots, \psi_i(\boldsymbol{x})) \end{array}$$

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- $F_i = \boldsymbol{\varphi}_{i-1}^{-1}(\boldsymbol{\varphi}_{i-1}(K)) \cap V$ critical fibers.
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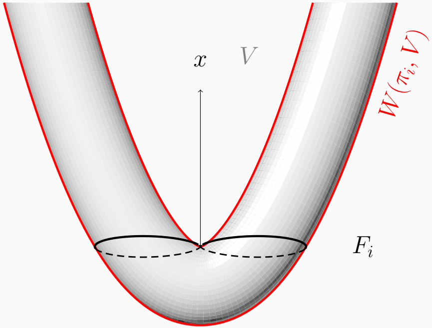
Roadmap property RM:

For all connected components C of V
$C \cap (F_i \cup W(\boldsymbol{\varphi}_i, V))$ is non-empty and connected



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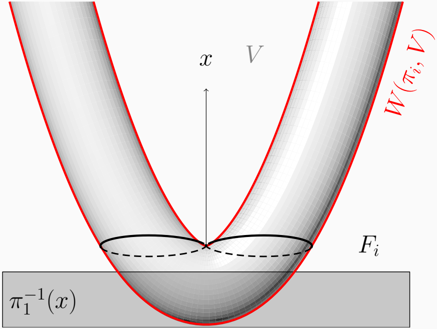

Morse theory

Two disjoint cases:
$x \in \boldsymbol{\varphi}_1^{-1}(K)$ or not

Sard's lemma

$\boldsymbol{\varphi}_1^{-1}(K)$ is finite

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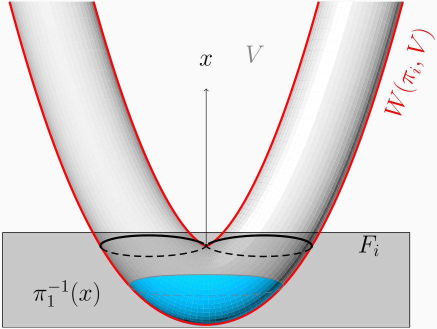
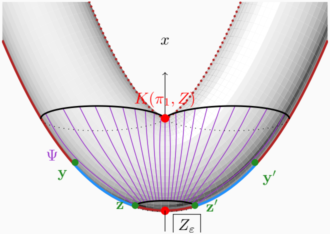


Thom's isotopy Lemma



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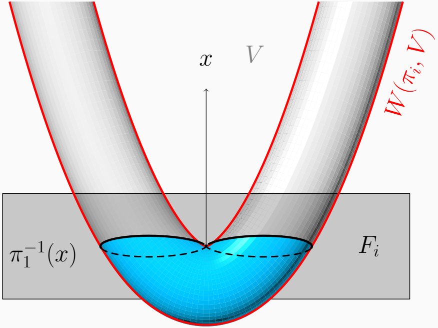
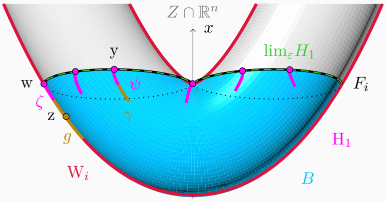


Algebraic Puiseux Series



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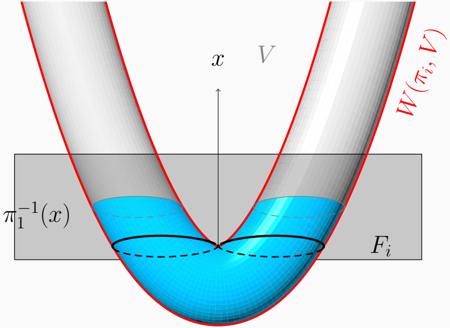

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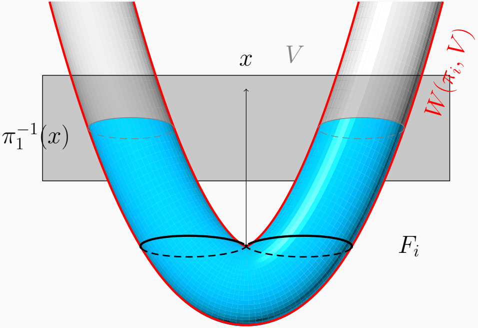

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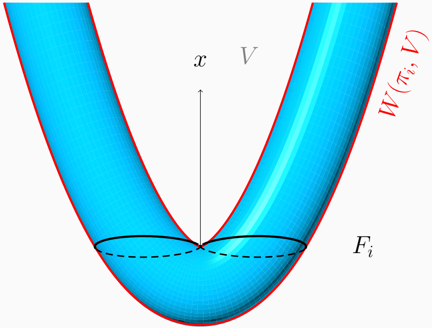

Genericity assumptions

Data

$\mathscr{C} \subset \mathbb{C}^n$ algebraic curve

$\pi_3 : \mathbb{C}^n \to \mathbb{C}^3$ projection on a **generic** 3D space

$\pi_2 : \mathbb{C}^n \to \mathbb{C}^2$ projection on a **generic** plane

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(H_1) $\pi_2 : \mathscr{C} \to \pi_2(\mathscr{C})$ is **birational**

(H_2) $\pi_3 : \mathscr{C} \to \pi_3(\mathscr{C})$ **bijective**

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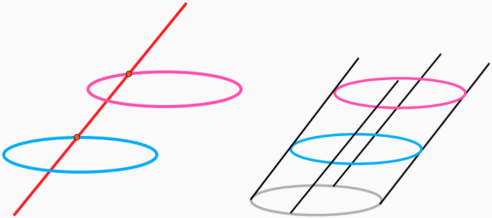
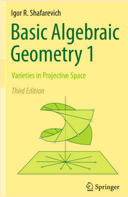


Secants are exceptional lines



[Shafarevich, '13]

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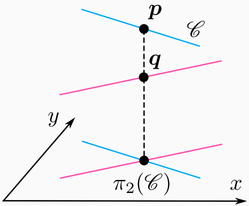
(H_3) Overlaps involve **at most two** points

(H_4) Overlaps introduce only **nodes**



Nodal apparent singularity

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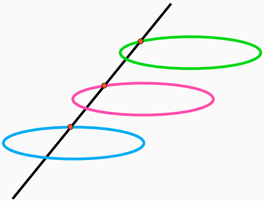

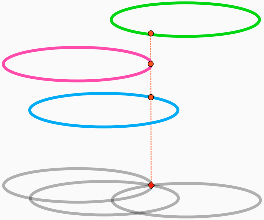

TriSecants are exceptional secants

Proof: Trisecant lemma for singular projective curves


TRISECANT LEMMA FOR NONEQUIDIMENSIONAL VARIETIES

J. Y. Kaminski, A. Kanel-Belov, and M. Teicher

[Kaminski Kanel-Belov Teicher; '08]

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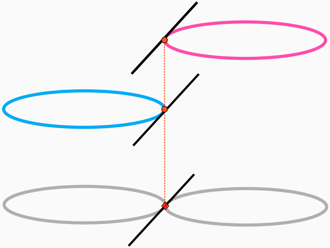
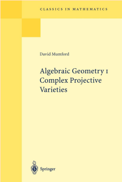
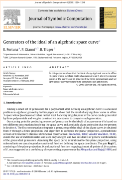


Secants with coplanar tangents are exceptional secants

Proof: Generalize results from literature



[Mumford; '76]



[Fortuna, Gianni Trager; '09]

Witness apparent singularities

- $\mathscr{R} = (\boldsymbol{\omega}, \boldsymbol{\rho_3}, \ldots, \rho_n) \subset \mathbb{Z}[x, y]$ encoding $\mathscr{C} \subset \mathbb{C}^n$ in generic position;
- $\mathcal{A}(x, y) = \partial^2_{x_2} \boldsymbol{\omega} \cdot \partial_{x_1} \boldsymbol{\rho_3} - \partial^2_{x_1 x_2} \boldsymbol{\omega} \cdot \partial_{x_2} \boldsymbol{\rho_3} \in \mathbb{Z}[x, y]$

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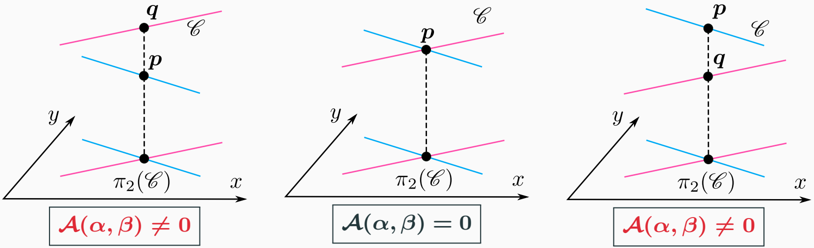
Proposition - Generalization of [El Kahoui; '08]

A **node** (α, β) is an **apparent singularity** if and only if $\mathcal{A}(\alpha, \beta) \neq 0$



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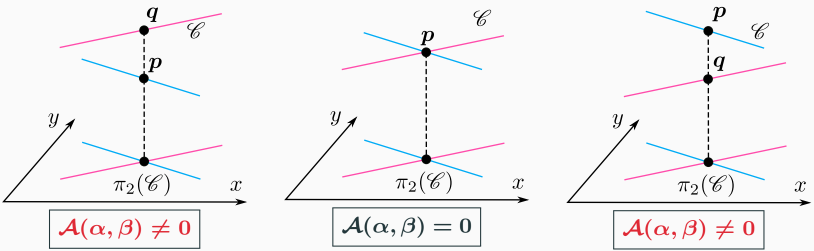


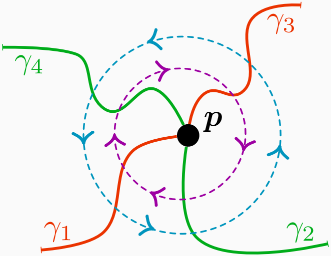
Computational aspect

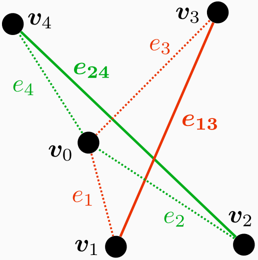
1. Non-vanishing can be tested using **gcd computations**
2. Gcd computations can be done modulo prime numbers

Lift connectivity

Recover connectivity ambiguity

At each vertex associated to an apparent singularities, operate two steps



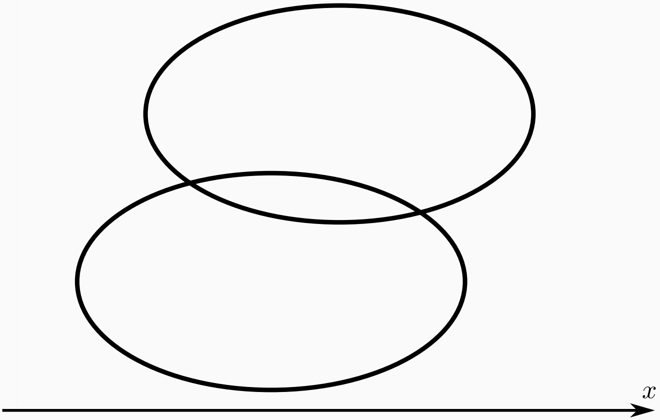


1st step

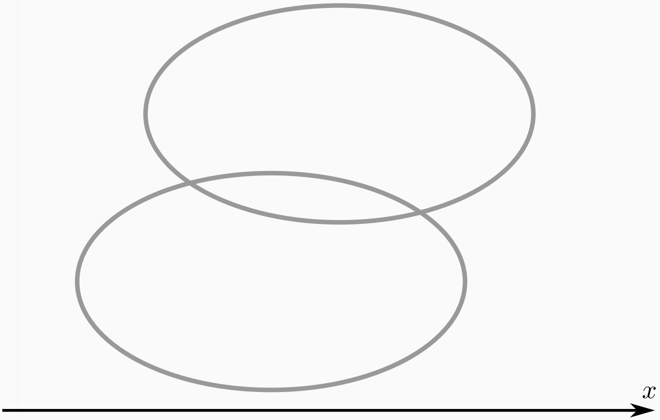
Identify opposite branches

2nd step

Modify the graph


x

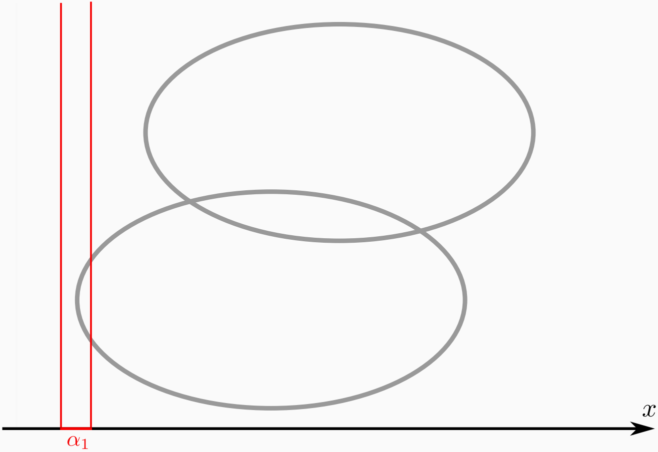
Computing the topology of plane curves



Cylindrical algebraic decomposition

Decompose the plane into cylinders where the topology of the curve can be computed

Computing the topology of plane curves



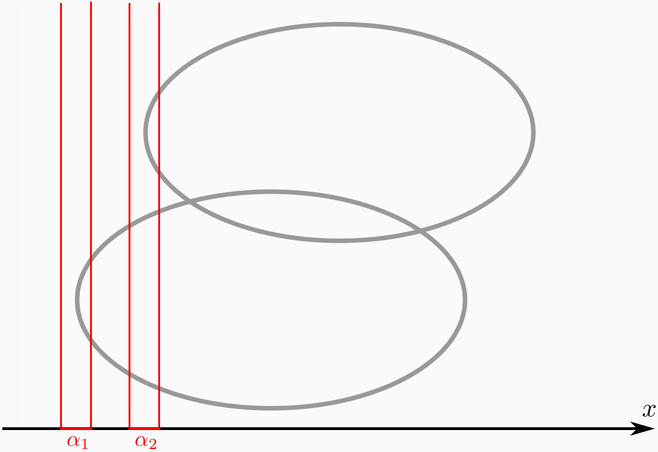
Cylindrical algebraic decomposition

Decompose the plane into cylinders where the topology of the curve can be computed

Morse theory

Topology changes at x-critical values

Computing the topology of plane curves

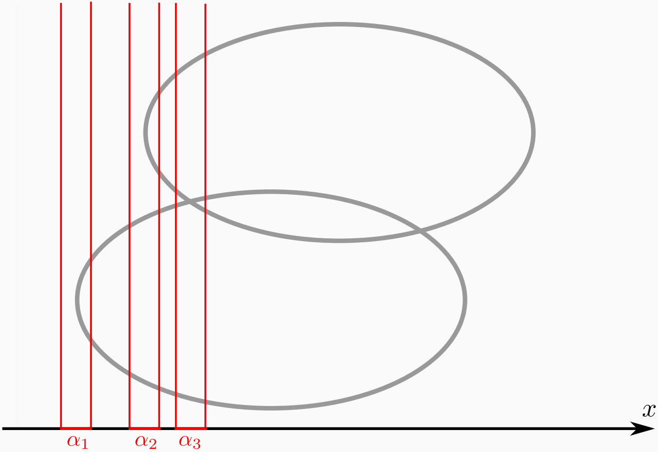


Isolating critical values

Isolation roots of the resultant of two bivariate polynomials

Complexity: $\tilde{O}(\delta^5(\delta + \tau))$ [Kobel, Sagraloff; '15] [D.Diatta, S.Diatta, Rouiller, Roy, Sagraloff; '22]

Computing the topology of plane curves



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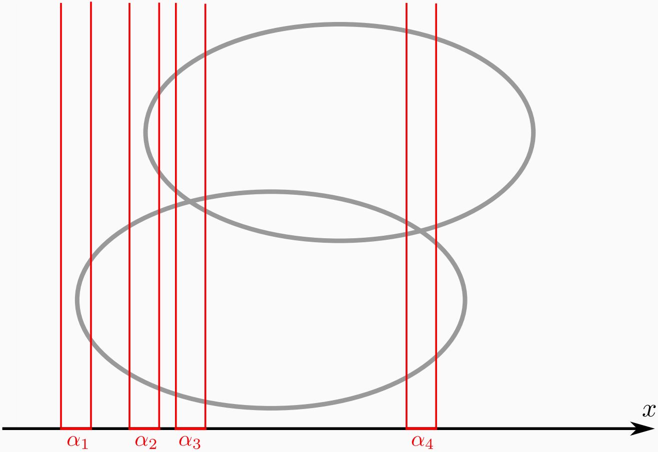
Complexity: $\tilde{O}(\delta^5(\delta + \tau))$

[Kobel, Sagraloff; '15]

[D.Diatta, S.Diatta, Rouiller, Roy, Sagraloff; '22]

Computing the topology of plane curves



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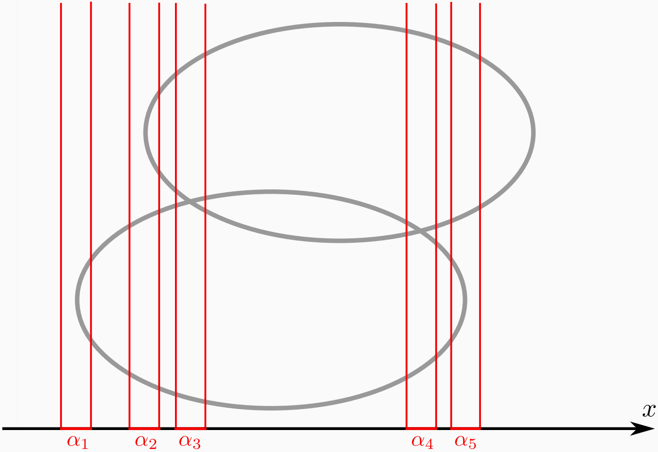

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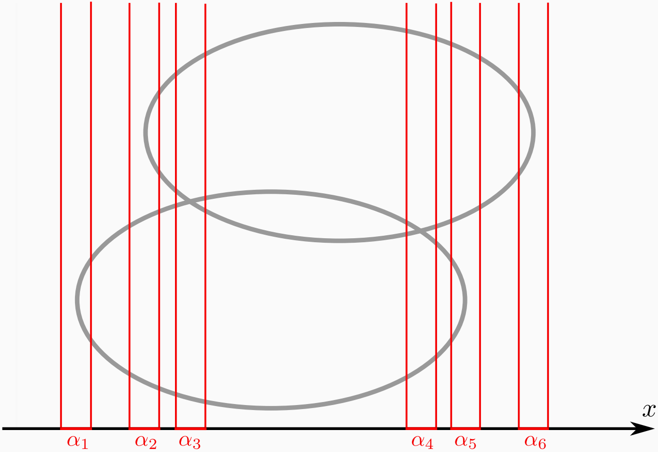

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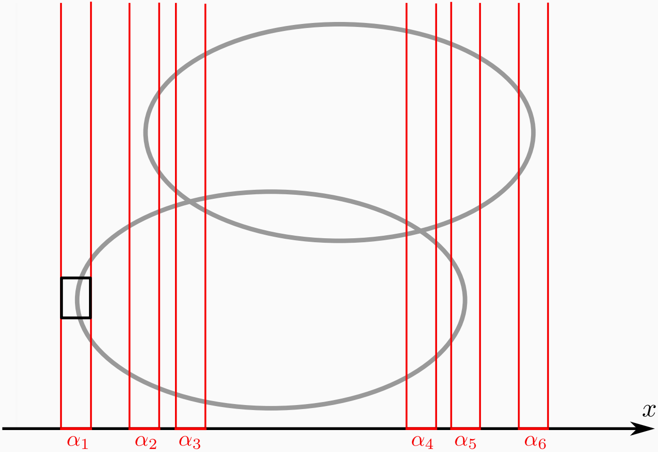

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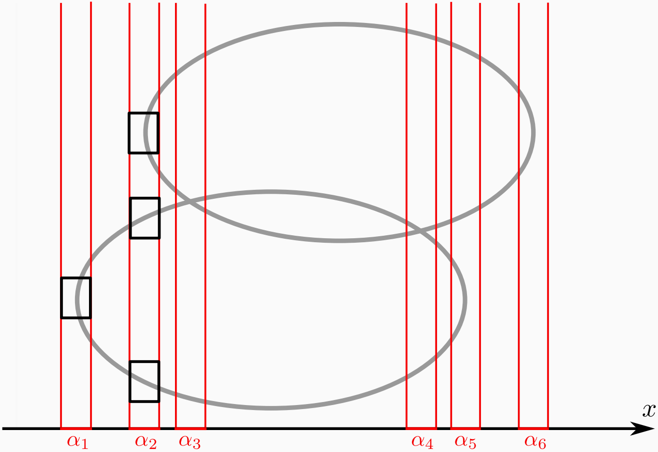

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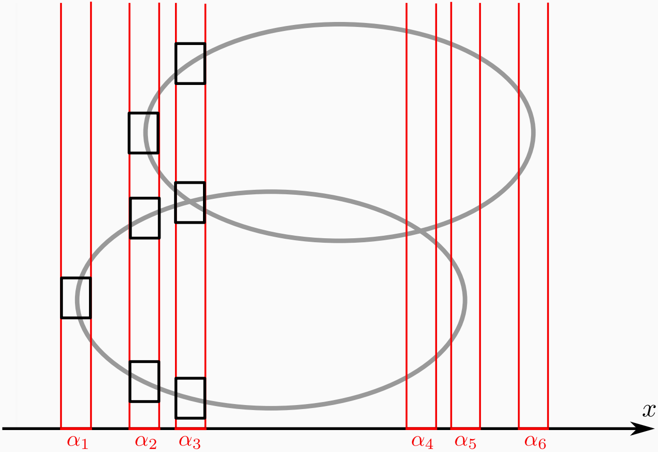

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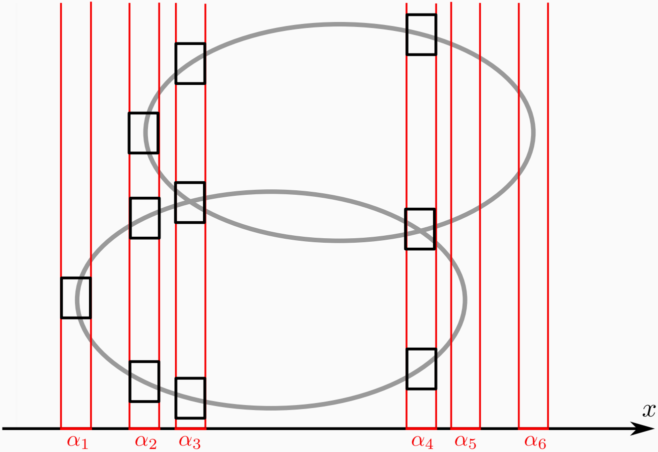

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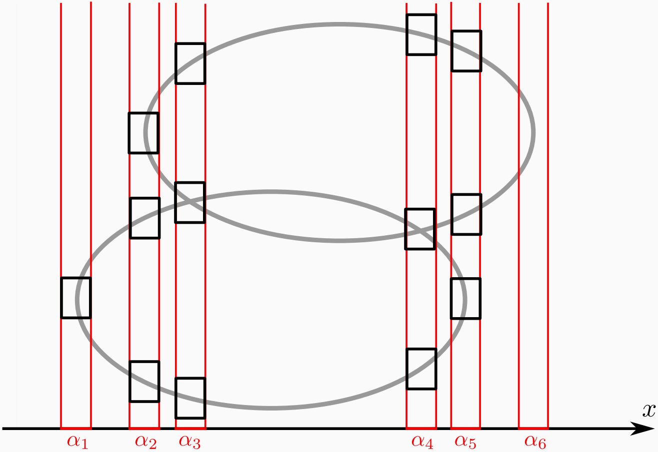

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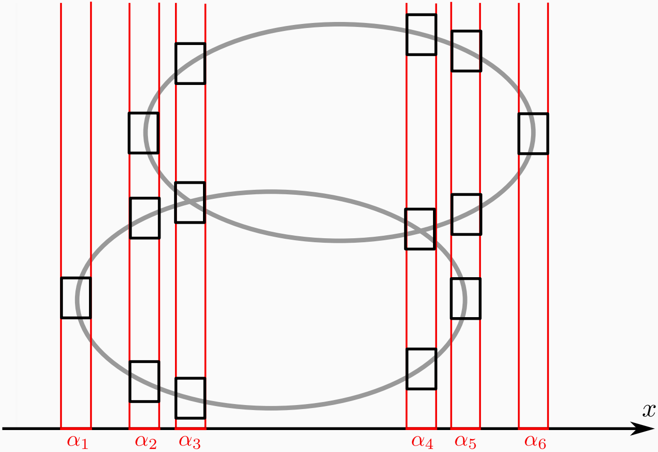

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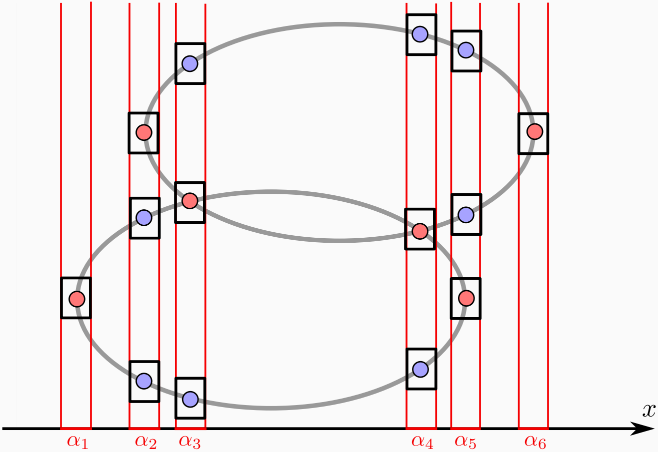

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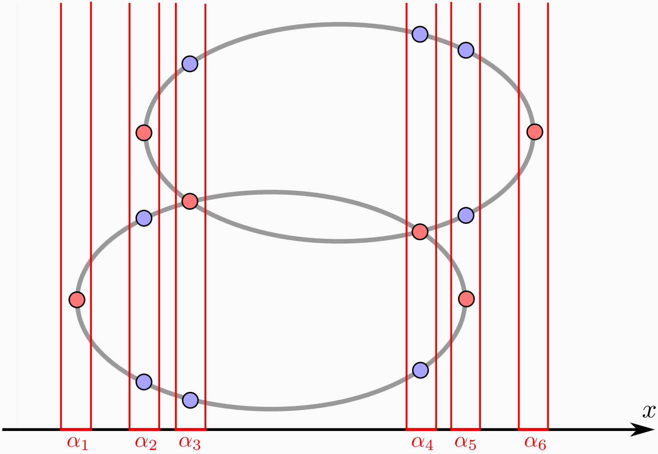
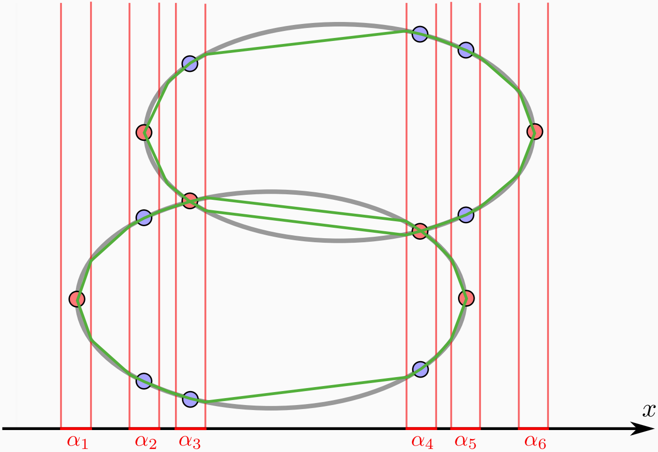

x
α_1
α_2
α_3
α_4
α_5
α_6


x
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Quantitative bounds on algebraic sets

Real algebraic sets

$$V_{\mathbb{R}} = \{f_1 = \cdots f_p = 0\} \subset \mathbb{R}^n$$
where
$$(f_1, \ldots, f_p) \subset \mathbb{R}[x_1, \ldots, x_n]$$

$\iff$

Real trace of algebraic sets

$$V_{\mathbb{R}} = V \cap \mathbb{R}^n$$
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Irreducible decomposition

$$V = V_1 \cup \cdots \cup V_M \qquad V_i \textit{ irreducible}$$

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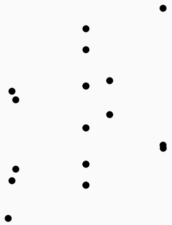

$$V = \{p_1, \ldots, p_{15}\}$$
$$\deg V = 15$$

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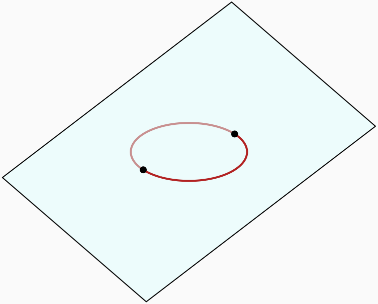

$\boldsymbol{V}\left(x^2 + y^2 - 1,\ z\right)$
$\Rightarrow \deg V = 2$

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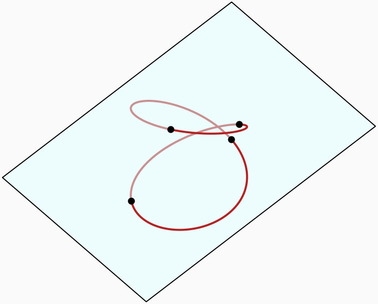
Bézout Bound

$$\deg V \leq \prod_{j=1}^{p} \deg f_j$$



$$\boldsymbol{V}\left(x^2 + y^2 - 1,\ 2z^2 - x - 1\right)$$
$$\Rightarrow \deg V = 4$$

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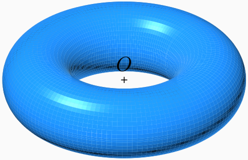

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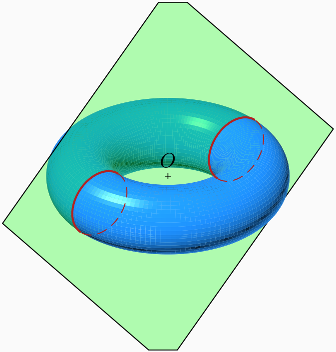

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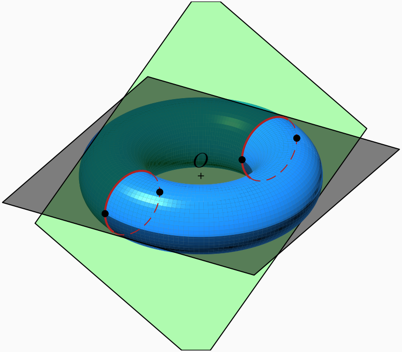

$$\boldsymbol{V}\left((x^2 + y^2 + z^2 + \alpha)^2 - \beta(x^2 + y^2)\right)$$
$$\Rightarrow \deg V = 4$$

Reduction

Consider $S = \{\boldsymbol{x} \in \boldsymbol{R}^n \mid f(\boldsymbol{x}) \neq 0\}$

Assumption 1: S is bounded. [Canny, 1988]

For $r > 0$ large enough,

$$\mathsf{RoadMap}\big(S \cap \overline{\mathcal{B}}(0, r)\big) = \mathsf{RoadMap}(S)$$

Assumption 2: S is an algebraic set [Canny, 1993]

For $\varepsilon > 0$ small enough,

$$\mathsf{Roadmap}\big(\{f \neq 0\} \cap \overline{\mathcal{B}}(0, r)\big) \rightarrow \mathsf{Roadmap}\big(\{f \geq \varepsilon\} \cap \overline{\mathcal{B}}(0, r)\big) \bigcup \mathsf{Roadmap}\big(\{f \leq -\varepsilon\} \cap \overline{\mathcal{B}}(0, r)\big)$$

Boundaries

Sufficient to compute the intersection of $S \cap \overline{\mathcal{B}}(0, r)$ with the roadmaps of

$$S_\varepsilon^+ = \boldsymbol{V}(f - \varepsilon), \quad S_{\varepsilon,r}^+ = \boldsymbol{V}(f - \varepsilon, ||\boldsymbol{x}||^2 - r), \quad S_r^+ = \boldsymbol{V}(||\boldsymbol{x}||^2 - r)$$

$$\text{and} \quad S_\varepsilon^- = \boldsymbol{V}(f + \varepsilon), \quad S_{\varepsilon,r}^- = \boldsymbol{V}(f + \varepsilon, ||\boldsymbol{x}||^2 - r), \quad S_r^- = \boldsymbol{V}(||\boldsymbol{x}||^2 - r).$$

Computation of critical loci

Critical points

$$\boldsymbol{x} \text{ critical point of } \pi_i \text{ on } V \iff \{\boldsymbol{x} \in \operatorname{reg}(V) \mid \pi_i(T_{\boldsymbol{x}}V) \neq \boldsymbol{C}^i\} = W^\circ(\pi_i, V)$$

An effective characterisation

$\boldsymbol{x}$ critical point of π_i on V

$J_i = \operatorname{Jac}(\boldsymbol{h}, [x_{i+1}, \ldots, x_n])$ where $\boldsymbol{h} \in \boldsymbol{I}(V) \subset \boldsymbol{R}[x_1, \ldots, x_n]$

(Lemma) $\downarrow$ $c = n - \dim(V)$

$\{\boldsymbol{x} \in V \mid \operatorname{rank} J_i(\boldsymbol{x}) < c\} \longrightarrow$ All c-*minors* of $J_i(\boldsymbol{x})$ vanish at $\boldsymbol{x}$

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Determinantal ideal

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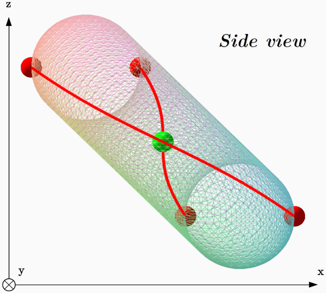

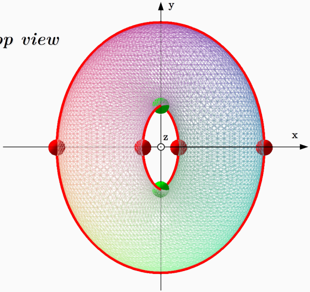

Torus of revolution axis directed by the vector $\vec{\boldsymbol{x}} + \vec{\boldsymbol{z}}$

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Determinantal ideal

Two kinds of critical points

$\boldsymbol{x}$ critical point of π_i on V

$\boldsymbol{x} \in W_2$ (polar variety)

↓

$T_{\boldsymbol{x}}W_2$ is normal to $\mathrm{Im}(\pi_1)$

⟶ $T_{\boldsymbol{x}}W_2 \subset T_{\boldsymbol{x}}V$ is normal to $\mathrm{Im}(\pi_1)$

OR

⟶ $T_{\boldsymbol{x}}W_2$ is normal to $\mathrm{Im}(\pi_2) \supset \mathrm{Im}(\pi_1)$

Splitting in two sets $\Longrightarrow$ Degree reduction

First results on the PUMA-type robot

Parameters

Parameters $(a_2, a_3, d_3, d_4, d_5) = (114, 40, 40, 104, 6)$ (Generic in in $\{1, \ldots, 128\}$)

Thresholds

$(\varepsilon, r) = (2^{-16}, 2^9)$

First step - computation of a parametrisation of critical locus over the algebraic sets

Alg. set	Dimension			Degree			Real points			Timings	
	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+	`msolve`	Maple
V	3	2	3	11	22	2				0.0 min	0.0 min
$K(1, V)$	0	0	0	400	934	2	88	116	2	4.8 min	84 min
$K_{\text{vert}}(2, V)$	0	0	0	354	924	0	8	66	0	5.3 min	49 min
$K(2, V)$	1	1	1	220	182	2				77 min	280 min

Library `msolve` `https://msolve.lip6.fr`

New library for solving zero-dimensional ideals.
Performances bring back the state-of-the art to the scope of laptops.

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First step - computation of a parametrisation of critical locus over the algebraic sets with `msolve`

Alg. set	Dimension			Degree			Real points			Timings (min.)		
	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+
V	3	2	3	11	22	2				0	0	0
$K(1, V)$	0	0	0	400	934	2	88	116	2	1.8	3.1	0
$K_{\text{vert}}(2, V)$	0	0	0	354	924	0	8	66	0	1.9	3.4	0
$K(2, V)$	1	1	1	220	182	2				108	39	0

Library `msolve` `https://msolve.lip6.fr`

New library for solving zero-dimensional ideals.

Performances bring back the state-of-the art to the scope of laptops.

First results on the PUMA-type robot

Parameters

Parameters $(a_2, a_3, d_3, d_4, d_5) = (114, 40, 40, 104, 6)$ (Generic in in $\{1, \ldots, 128\}$)

Thresholds

$(\varepsilon, r) = (2^{-16}, 2^9)$

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Alg. set	Dimension			Degree			Real points			Timings (min.)		
	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+
V	3	2	3	11	22	2				0	0	0
$K(1, V)$	0	0	0	400	934	2	88	116	2	1.8	3.1	0
$K_{\text{vert}}(2, V)$	0	0	0	354	924	0	8	66	0	1.9	3.4	0
$K(2, V)$	1	1	1	220	182	2				108	39	0

Recursive step - critical locus over fibers of S_ε^+.

There are $88 + 8 = \mathbf{96}$ fibers.

Alg. set	Dimension	Degree	Real points	Timings: One fiber	Timings: All fibers
F_ε	2	7		3 s	4.75 min
$K(1, F_\varepsilon)$	0	38	14	2 s	3.2 min
$K_{\text{vert}}(2, F_\varepsilon)$	0	0	0	0 s	0.0 min
$K(2, F_\varepsilon)$	1	21		3 s	4.8 min

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Parameters

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First step - computation of a parametrisation of critical locus over the algebraic sets with `msolve`

Alg. set	Dimension			Degree			Real points			Timings (min.)		
	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+	S_ε^+	$S_{\varepsilon,r}^+$	S_r^+
V	3	2	3	11	22	2				0	0	0
$K(1, V)$	0	0	0	400	934	2	88	116	2	1.8	3.1	0
$K_{\text{vert}}(2, V)$	0	0	0	354	924	0	8	66	0	1.9	3.4	0
$K(2, V)$	1	1	1	220	182	2				108	39	0

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Roadmap

Degree: **8168**

Time: **3h22**

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Hyperlinks

Cuspidality

Slides: Cusp definition | Cusp resolution

Bonus: Thom's | Correction | Algorithm | Application | Sample Points | Connectivity queries

Roadmap

Slides: Canny's strategy | Roadmap state-of-the-art | Genericity assumptions | Algorithm

Bonus: Proof of the new connectivity result

PUMA robot

Bonus: Reduction to alg. sets | Splitting critical loci | Computational details

Curves

Slides: Rational Parametrization | State-of-the-art | Algorithm

Bonus: Genericity assumptions | App sing. identification | Node resolution | Plane topology

Misc

Slides: Main contributions | Perspectives

Bonus: Quantitative bounds on alg. sets