Laws of concurrent system design

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The Laws: Summary

- What are they?
- What do they mean?
- Are they useful?
- Are they true?
- Are they beautiful?

1. The Laws

are algebraic equations like

$$2xpxq + qxq \leq (p+q) \times (p+q)$$

Variables p, q, r, …

- stand for specifications/designs/programs
  - describing all behaviours of a computer
  - that are desired/planned/actual
  - when the program is executed
- a single behaviour is recorded
  - as a set of events,
  - occurring inside or near a computer system
  - while it is executing a program

Three operators

• then	;	sequential composition
• with	||	concurrent composition
• skip	I	does nothing

Their intended meaning

- then ; sequential composition
- with || concurrent composition
- skip I does nothing
- p;q describes the behaviour resulting from

  execution of p till completion

  followed by execution of q
- p||q describes their concurrent execution

  p and q start together and finish together

Five Axioms

- assoc p;(q;r) = (p;q);r (also ||)
- comm p||q = q||p
- unit p||I = p = I||p (also ;)

Reversibility

- assoc p;(q;r) = (p;q);r (also ||)
- comm p||q = q||p
- unit p||I = p = I||p (also ;)

- swapping the order of operands of ; (or of ||) translates each axiom into itself.
- and each proof into a swapped proof.

Duality

- Metatheorem: (theorems for free)

When a theorem is translated by reversing the operands of **all** ;s the result is also a theorem.

- (same for **all** ||s)

- Analogy: many laws of physics remain true when the direction of time is reversed.

Refinement: p => q

- means every execution described by p
  is also described by q
- in other words,
  - program p is more predictable and more controllable than program q
  - program p meets spec q
  - design p conforms to design q

Axiom

- => is a partial order
  - reflexive p => p
  - transitive if p => q & q => r then p => r
- swapping the operands of => translates each axiom into itself
- justifies duality by order reversal
  - if p => q is a theorem proved from these axioms
    so is q => p
- (later axioms will violate this duality)

Monotonicity

- Definition: an operator • is monotonic if
  - p => q implies p•r => q•r
    & r•p => r•q

- Axioms: ; and || are monotonic

- In a theorem, we can replace any subterm
  of a term on the left (right) of =>
  by one that is more (less) refined

Monotonicity

- Metatheorem :

Let p => q be a theorem

Let F be a formula containing p.

Let F' be a modification of F

that just replaces an occurrence of p by q

Then F => F' is also a theorem

Exchange Axiom

- (p||q) ; (p'||q') => (p;p') || (q;q')

- LHS describes certain interleavings of RHS
  - those where the two RHS ;s are synchronised
- implemented by interleaving p with q
- followed by an interleaving of q with q'

Exchange Axiom

- (p||q) ; (p'||q') => (p;p') || (q;q')
- Theorem (frame): (p||q) ; q' => p||(q;q')
  - Proof: substitute I for p' in exchange axiom
- Theorem: p;q' => p||q'
  - Proof: substitute I for q

- This axiom is self-dual by time-reversal
  - but not by order-reversal

2. Applications

to Hoare logic

and to Milner transitions

The laws are useful

- for proof of correctness of programs/designs
  - by means of Hoare logic
  - (extended by concurrent separation logic)
  - to describe the structure of proofs
- for design/proof of implementations
  - using Milner transitions
  - (extended by sequential composition)
  - to describe the steps of execution.

The Hoare triple

- Definition: {p} q {r} = p;q => r
  - If p describes what has happened so far,
  - and then q is executed to completion,
  - the overall execution will satisfy r.

- Example: p and r may be ‘assertions’,
  - describing all executions that leave the machine in a state satisfying a given Boolean predicate.

The rule of composition

- Definition: {p} q {r} = p;q => r
- Theorem:

$$\frac{\{p\}\ q\ \{s\} \qquad \{s\}\ q'\ \{r\}}{\{p\}\ q;q'\ \{r\}}$$

Proof

- Definition: {p} q {r} = p;q => r
- expanding the definition:

$$\frac{p;q \Rightarrow s \qquad s;q' \Rightarrow r}{p;q;q' \Rightarrow r}$$

because ; is monotonic and associative

The rule of consequence

- Theorem

$$\frac{p' \Rightarrow p \qquad \{p\}\ q\ \{r\} \qquad r \Rightarrow r'}{\{p'\}\ q\ \{r'\}}$$

Proof: monotonicity and transitivity

Modularity rule for ||

- in concurrent separation logic

$$\frac{\{p\}\ q\ \{r\} \qquad\qquad \{p'\}\ q'\ \{r'\}}{\{p||p'\}\ (q||q')\ \{r||r'\}}$$

  - permits modular proof of concurrent programs.

- it is **equivalent** to the exchange law

Modularity rule implies Exchange law

- By reflexivity: p;q => p;q and p’;q’ => p’;q’
- take these as antecedents of modularity rule
  - replacing r, r’ by p;q and p;q’ ,
- After the same substitution, the conclusion of the modularity rule gives

  (p||p’) ; (q||q’) => (p;q) || (p’;q’)

  - which is the Exchange law

Exchange law implies modularity

- Assume: p;q => r and p’;q’ => r’
- monotonicity of || gives

  (p;q) || (p’;q’) => r|| r’
- the Exchange law says

  (p||p’) ; (q||q’) => (p;q) || (p’;q’)
- by transitivity:

  (p||p’) ; (q||q’) => r||r’

which is the conclusion of the modularity rule

Frame Rules

$$\frac{\{p\}\ q\ \{r\}}{\{p \,||\, f\}\ q\ \{r \,||\, f\}}$$

- adapts a triple to a concurrent environment f
- proof: from frame theorem

$$\frac{\{p\}\ q\ \{r\}}{\{f;p\}\ q\ \{f;r\}}$$

- proof: mon, assoc of ;

The Milner triple: r - q -> p

- defined as q;p => r
  - (the time reversal of {p} q {r})
- r may be executed by first executing q
  - with p as continuation for later execution.
  - maybe there are other ways of executing r

- Tautology: (q ; p) – q -> p (CCS)
- Proof: from reflexivity: q;p => q;p

Technical Objection

- Originally, Hoare restricted q to be a program, and p , r to be state descriptions
- Originally, Milner restricted p and r to be programs, and q to be an atomic action.
- These restrictions are useful in application.
- And so is their removal in theory
  - (provided that the axioms are still consistent).

Sequential composition

$$\frac{\{p\}\ q'\ \{s\} \qquad\qquad \{s\}\ q\ \{r\}}{\{p\}\ q';q\ \{r\}}$$

$$\frac{r\ \text{–}q\text{->}\ s \qquad\qquad s\ \text{–}q'\text{->}\ p}{r\ \text{–}(q;q')\text{->}\ p}$$

Proof: by time-reversal of the Hoare rule

Concurrency in CCS

$$\frac{r \xrightarrow{p} q \qquad r' \xrightarrow{p'} q'}{(r||r') \xrightarrow{(p||p')} (q||q')}$$

Proof: by time-reversal of the modularity rule

- In Milner's CCS, the rule is applied
  only if p and p' are synchronised, e.g.,
  input and output on the same channel.

Frame Rules

$$\frac{r \xrightarrow{q} p}{(r \,||\, f) \xrightarrow{q} (p \,||\, f)}$$

– a step q possible for a single thread r is still possible when r is executed concurrently with f

$$\frac{r \xrightarrow{q} p}{(r;f) \xrightarrow{q} (p;f)}$$

– operational definition of ;

The internal step

- r -> p $=_{def.}$ p => r
  - (the order reversal of refinement)
- implementation may make a refinement step
  - reducing the range of subsequent behaviours

Rule of consequence

- $\dfrac{p \Rightarrow p' \qquad \{p'\}\ q\ \{r'\} \qquad r' \Rightarrow r}{\{p\}\ q\ \{r\}}$
- $\dfrac{r \to r' \qquad r' \xrightarrow{q} p' \qquad p' \to p}{r \xrightarrow{q} p}$

- Each rule is the dual of the other
  - by order reversal and time reversal

Axioms proved from calculi

from Hoare

- $p ; (q \bigvee r) \Rightarrow p;q \bigvee p;r$
- $p;r \bigvee q;r \Rightarrow (p \bigvee q) ; r$

from Milner

- $(p \bigvee q) ; r \Rightarrow (p;r) \bigvee (q;r)$
- $p;q \bigvee p;r \Rightarrow p ; (q \bigvee r)$

from both

- $p ; (q;r) \Rightarrow (p;q) ; r$
- $(p;q) ; r \Rightarrow p ; (q;r)$
- exchange law

Message

- **Both the Hoare and Milner rules are derived from the same algebra of programming.**
- **The algebra is simpler than each of the calculi,**
- **and stronger than both of them combined.**
- **Deductive and operational semantics are mutually consistent, provided the laws are true**

3. The laws are true

of a realistic mathematical model
of real program behaviour

Behaviours

- are sets of events
  - occurring in and around a computer
  - that is executing a program

- Let **Ev** be the set of all occurrences
  - of all such events
  - that ever were, or ever could be

Happens before (→)

- Let e, f, g ϵ **Ev** (sets of event occurrences).
- e → f is intended to mean (your choice of) :
  - "the occurrence e is an immediate and necessary cause of the occurrence f "
  - "the occurrence f directly depends (depended) on the occurrence e "
  - "e happens before f " "f happens after e "

Examples: software

- n^{th} output → n^{th} input (on a reliable channel)
- n^{th} V (acquire) → n^{th} P (release)

  (on an exclusion semaphore)
- n^{th} assignment → read of the n^{th} value assigned

  (to a variable in memory)
- read of n^{th} value → $(n + 1)^{st}$ assignment

  (in strong memory)

Precedes/follows

- Define $\leq$ as $(\rightarrow)^*$
  - the reflexive transitive closure of $\rightarrow$
  - Define $\geq$ as $\leq^\circ$ (the converse of $\leq$)
- Examples:
  - allocation of a resource $\leq$ every use of it
  - disposal of a resource $\geq$ every use of it

Interpretations

- $e \leq f$ & $f \leq e$ means
  - e and f are (parts of) the same atomic action
- **not** $e \leq f$ & **not** $f \leq e$ means
  - e and f are independent of each other
  - their executions may overlap in time,
  - or one may complete before the other starts

Cartesian product

- Let $p, q, r \subseteq$ **Ev**
  - behaviours are sets of event occurrences
- Define $p \times q = \{(e,f) \mid e \in p \ \& \ f \in q\}$
  - the Cartesian product
- Theorem: $p \times (q \cup r) = p \times q \cup p \times r$

$$(q \cup r) \times p = q \times p \cup r \times p$$

Composition

- Let $p, q, r \subseteq$ **Ev** (behaviours)
- Let **seq** $\subseteq$ **Ev** $\times$ **Ev** (arbitrary relation)
- Define $p;q = p \cup q$ **if** $p \times q \subseteq$ **seq**

  & p, q are defined

  – and is undefined otherwise
- Define $p \sqsubseteq q$ as $p = q$ or p is undefined

Theorem: ; is monotonic wrto $\leq$

Theorem: $(p ; q) ; r = p ; (q ; r)$

- Proof: when they are both defined, each side is equal to $(p \cup q \cup r)$.

- We still need to prove that LHS is defined iff ant RHS is defined.

Theorem: $(p ; q) ; r = p ; (q ; r)$

LHS is defined iff (by definition of ;)

$p \times q \subseteq \mathbf{seq} \;\&\; (p \cup q) \times r \subseteq \mathbf{seq}$

iff $p \times q \subseteq \mathbf{seq} \;\&\; p \times r \subseteq \mathbf{seq} \;\&\; q \times r \subseteq \mathbf{seq}$ (*)

iff $p \times (q \cup r) \subseteq \mathbf{seq} \;\&\; q \times r \subseteq \mathbf{seq}$ (*)

iff RHS is defined

*(by $\times$ distrib $\cup$)

Sequential composition (strong)

- Define **seq** = $\underline{\leq}$
- Then p;q is (strong) sequential composition
- means that p must finish before q starts
  - every event in p comes before every event in p
- Example: **Ev** is **NN** $\underline{\leq}$ is numerical <
  - {1, 7, 19} ; {21, 32} = {1, 7, 19, 21, 32}
  - {1, 7, 19} ; {19, 32} is undefined

Sequential composition (weak)

- Define **seq** = **not** $\geq$
- Then p;q is (weak) sequential composition
- means that p can finish before q starts
  - no event in q comes before any event in p
  - but q can often start before end of p,

  provided the exchanged events are independent.

Concurrent Composition

Define **par** = **Ev x Ev**

Note: **seq** $\subseteq$ **par** = **par**$^\circ$ (converse)

Theorem: pxq $\subseteq$ **par**

Define p||q = p $\cup$ q

Theorem: || is associative and commutative.

and satisfies exchange law with ; (weak)

Examples

- Example: **Ev** is **NN**
  - {1, 7, 19} ; {21, 32} = {1, 7, 19, 21, 32}
  - {1, 7, 19} ; {19, 32} is undefined
  - {1, 7, 19}||{3, 10, 32} = {1, 3, 7, 10, 19, 32}

(q || q') ; (r || r') => (q ; r) || (q' ; r')

- Proof: when LHS is defined, it equals RHS

$$q \cup r \cup q' \cup r'$$

$(q \,||\, q') ; (r \,||\, r') \Rightarrow (q ; r) \,||\, (q' ; r')$

LHS defined **iff** $q \times q' \subseteq$ **par** & $r \times r' \subseteq$ **par**

& $(q \cup q') \times (r \cup r') \subseteq$ **seq**

implies $q' \times r' \cup q \times r \cup q' \times r \cup q \times r' \subseteq$ **seq**

implies $q' \times r' \subseteq$ **seq** & $q \times r \subseteq$ **seq**

& $(q \cup r) \times (q' \cup r') \subseteq$ **par**

implies RHS defined.

4. The laws are useful

Tools for Software Engineering

Verification

Compilation

Testing

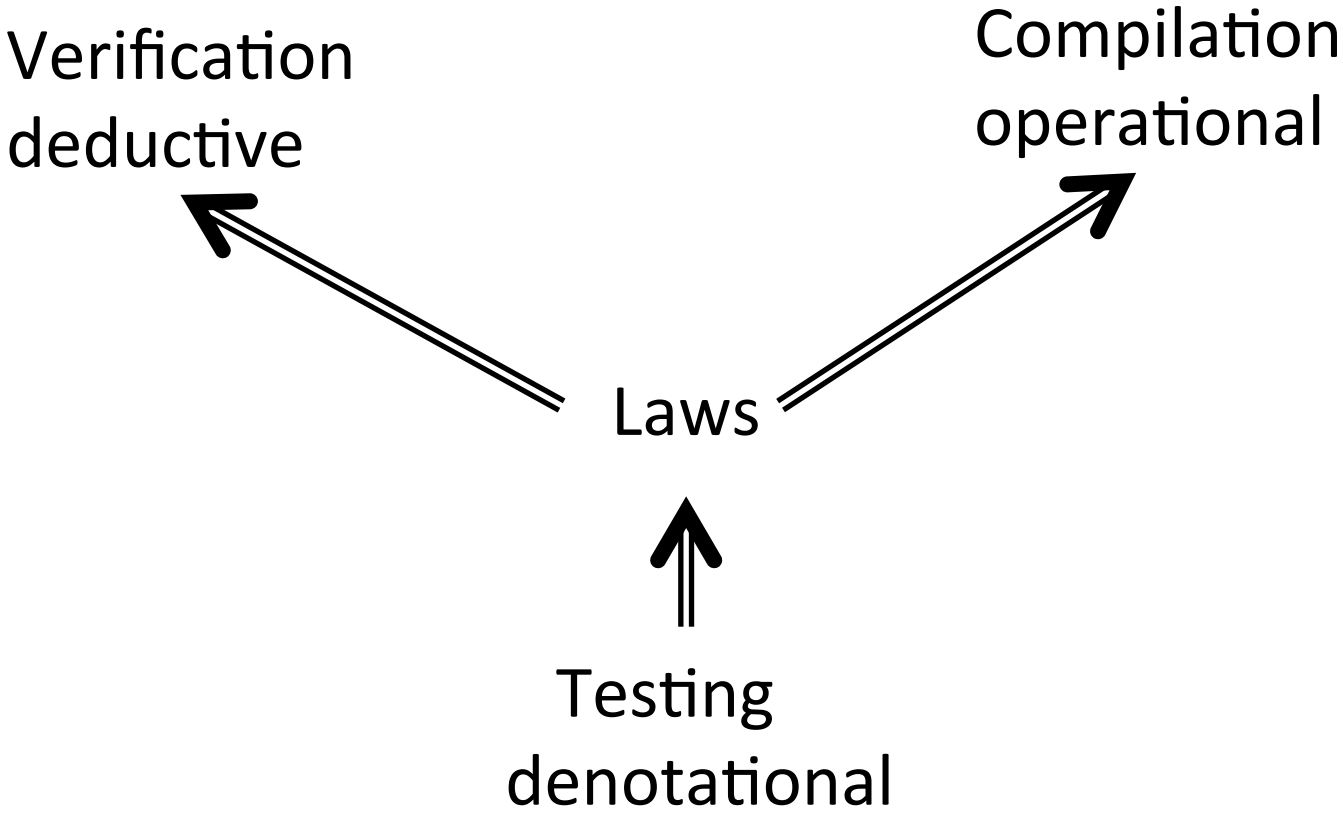
based on semantics

Verification
deductive (Hoare)

Compilation
operational (Milner)

Testing
denotational (Scott)

Laws prove consistency



5. Conclusion

The Laws

- The laws are useful
  - they shorten formulae, theorems, proofs
  - they prove consistency of proof rules with the implementation
- The laws are true
  - of specifications, designs, products
  - hardware/software/the real world
- The laws are beautiful

Isaac Newton

Communication with Richard Gregory (1694)

"Our specious algebra [the infinitesimal calculus] is fit enough to find out [is ok as a heuristic], but entirely unfit to consign to writing and commit to posterity."

Bertrand Russell

- The method of postulation has many advantages. They are the same as the advantages of theft over honest toil.

Introduction to Mathematical Philosophy.

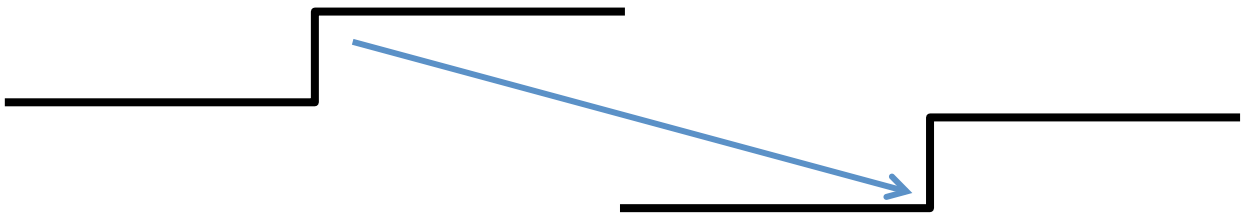
Gottfried Leibnitz

- Calculemus.

Examples: hardware

- a rising edge → next falling edge on same wire

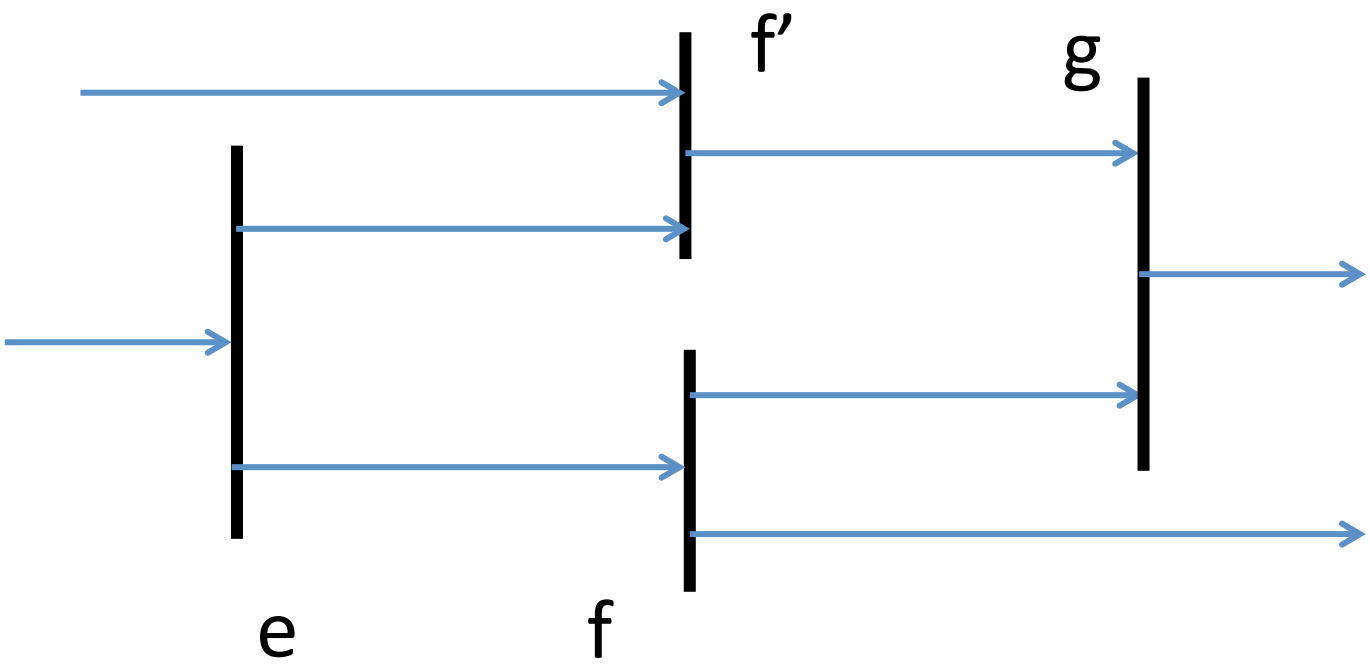
- a rising edge → rising edge on another wire



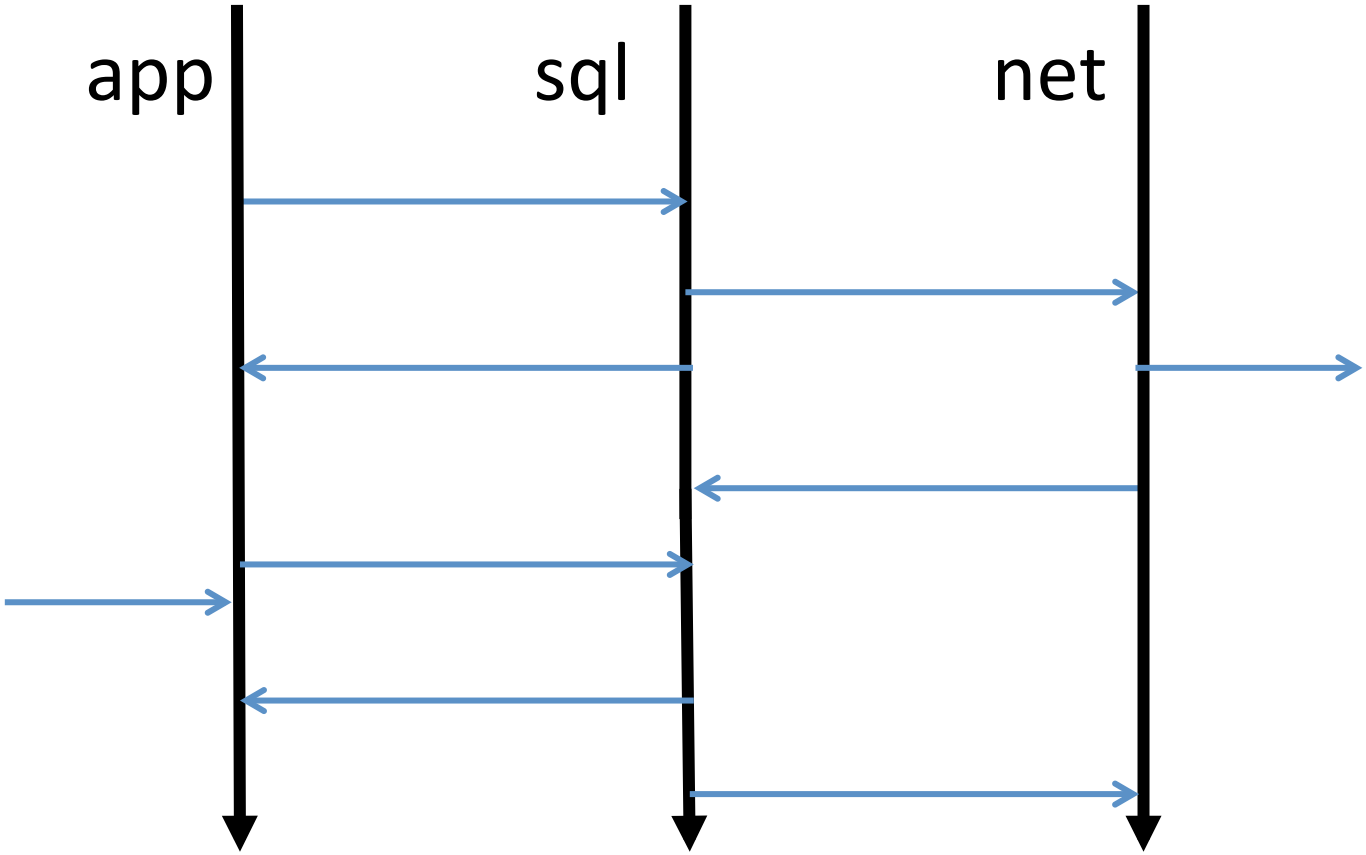
Example: Petri nets

e → f' & f' → g

e → f & f → g



Message sequence chart



Additional operators

- p \/ q describes all traces of p and all of q
  - describes options in design
  - choice (non-determinism) in execution
- p /\ q describes all traces of both p and q
  - conjunction of requirements (aspects) in design
  - lock-step concurrency in execution

Axioms

- \/ is the disjunction and /\ is the conjunction of a Boolean Algebra (even with negation).
- Axiom: ; and || distribute through \/
  - which validates reasoning by cases
  - and implementation by non-deterministic selection

Choice

- $$\frac{\{p\}\ q\ \{r\} \qquad \{p\}\ q'\ \{r\}}{\{p\}\ (q \vee q')\ \{r\}}$$
  - both choices must be correct
  - proof: distribution of ; through $\vee$

$$\frac{r \xrightarrow{q} p}{(r \vee r') \xrightarrow{q} p}$$

  - only one of the alternatives is executed
  - proof: $r \Rightarrow r \vee r'$

Axioms proved from calculi

from Hoare

- $p ; (q \bigvee r) \Rightarrow p;q \bigvee p;r$
- $p;r \bigvee q;r \Rightarrow (p \bigvee q) ; r$

from Milner

- $(p \bigvee q) ; r \Rightarrow (p;r) \bigvee (q;r)$
- $p;q \bigvee p;r \Rightarrow p ; (q \bigvee r)$

from both

- $p ; (q;r) \Rightarrow (p;q) ; r$
- $(p;q) ; r \Rightarrow p ; (q;r)$
- exchange law