Tracing the Topics in *Les Réseaux* (*ou Graphes*)

An Annotated Translation with Commentaries

Martin Charles Golumbic

University of Haifa, Israel

André Sainte-Laguë's 65-page monograph (1926)

An historical *snapshot* of graph theory at that time when many concepts were still in their infancy, yet others were highly developed.

Prophetically, Sainte-Laguë understood that *applications* would play a future role.

This lecture formally launches our new book

The Zeroth Book of Graph Theory:

An Annotated Translation of

Les Réseaux (ou Graphes)–André Sainte-Laguë (1926)

Springer (2021), xii+120 pp.

The term "the 0th book" came from Harald Gropp (1996)

I will trace some of the topics as they have evolved, especially as influenced by computing and informatics, including my own work in algorithmic graph theory.

Interspersed with the combinatorics, I will give stories and glimpses into the fascinating mathematical and non-mathematical career of André Sainte-Laguë.

MÉMORIAL

DES

SCIENCES MATHÉMATIQUES

PUBLIÉ SOUS LE PATRONAGE DE

L'ACADÉMIE DES SCIENCES DE PARIS

DES ACADÉMIES DE BELGRADE, BRUXELLES, BUCAREST, COÏMBRE, CRACOVIE, KIEW,
MADRID, PRAGUE, ROME, STOCKHOLM (FONDATION MITTAG-LEFFLER), ETC.,
DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE, AVEC LA COLLABORATION DE NOMBREUX SAVANTS.

DIRECTEUR :

Henri VILLAT
Correspondant de l'Académie des Sciences de Paris,
Professeur à l'Université de Strasbourg.

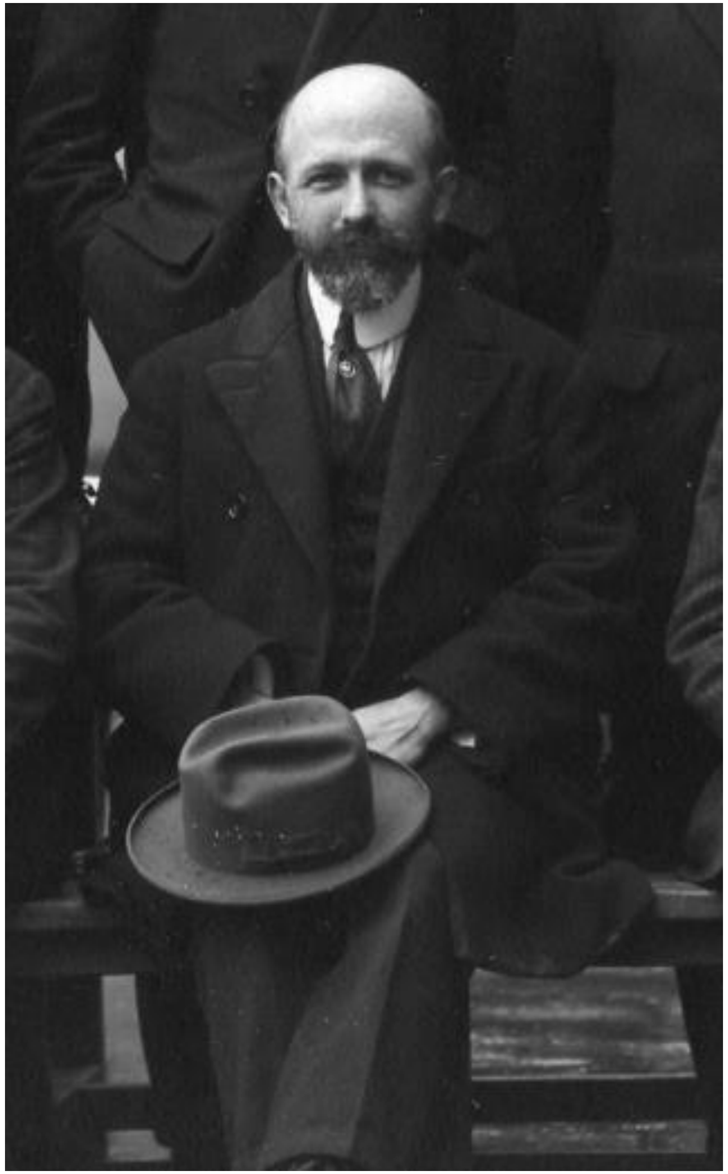
FASCICULE XVIII

Les Réseaux (ou graphes)

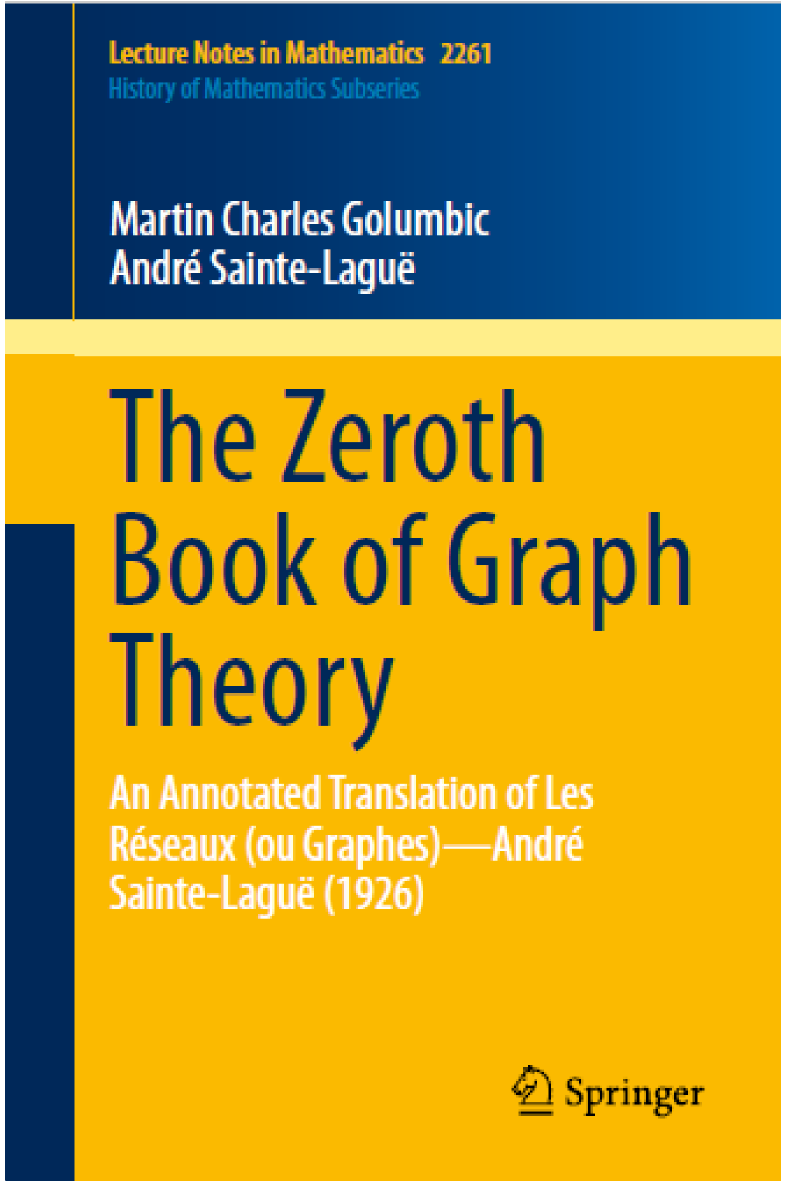
PAR M. A. SAINTE-LAGUË
Professeur au Lycée Carnot.

PARIS
GAUTHIER-VILLARS ET C^ie, ÉDITEURS
LIBRAIRES DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE
Quai des Grands-Augustins, 55

1926



André Sainte-Laguë among his students at the Lycée Janson-de-Sailly (Paris) in 1924



Contents

André Sainte-Laguë

- Perhaps best known historically for his method of *parliamentary seat allocation*, which he published in 1910, when he was 28 years old.
- Known today as the *Webster–Sainte-Laguë Method*, proposed in 1832 by the American statesman, Daniel Webster – it was first adopted in 1842 for the U.S. House of Representatives.
- The method is still used today in many countries around the world.

Born on April 20, 1882, in the village of Saint Martin Curton in southwest France, he graduated from the École Normale Supérieure in 1906, and taught high-school mathematics until the outbreak of World War I.

He was mobilized into the infantry at the front for two years, receiving several medals. Wounded three times, Sainte-Laguë wrote that

he took advantage of 'trench recreation time'
and his stays in military hospitals,
to pursue mathematical research on graphs and topology.

André Sainte-Laguë

From 1917 to 1919, Sainte-Laguë worked on long-range shell studies in the Department of Inventions and at the laboratories at the École Normale Supérieure.

After World War I, Sainte-Laguë returned to teaching mathematics at the Lycée Pasteur de Neuilly-sur-Seine, while continuing his research and completing his doctorate in June 1924.

Listed on his dissertation committee are the well-known French mathematicians Émile Picard (President), Émile Borel and Paul Montel (Examiners).

André Sainte-Laguë

Jérôme Chastenet de Géry [234] has written, (translated from the original French):

His classes enjoyed considerable success, unequaled until then. One of his lectures had up to 2500 listeners, forcing him to give it three times in the great 900-seat Paul Painlevé amphitheater at CNAM.

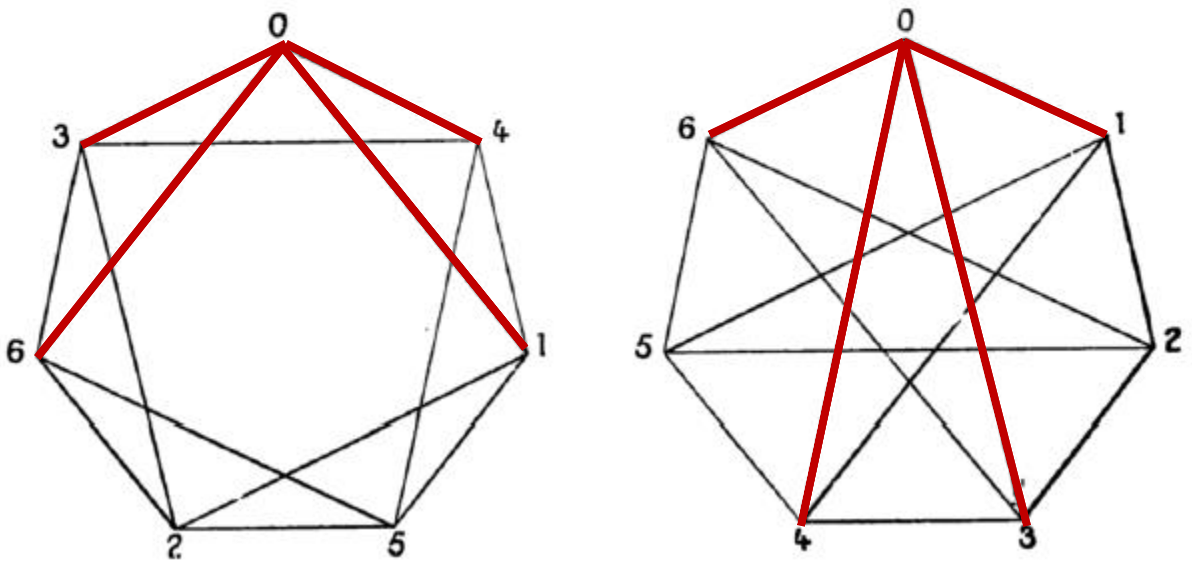
His warm and loud voice filled the room, and his lectures were lively, fast, and clear.

` **From 1928 onwards, he also used films for his geometry lessons.**

I. Introduction and definitions

A *network* or *graph* is a set of ***vertices*** or *points*

joined by ***edges*** or *lines* connecting pairs of vertices.



Two drawings of the **same** graph.

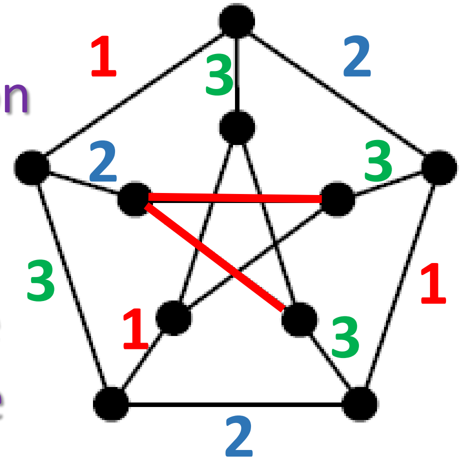
What matters is only whether two vertices A and B are connected by an edge.

These two graphs are called ***regular*** – each vertex has the *same number of neighbors*.

Regular graphs is the topic of Chapter IV.

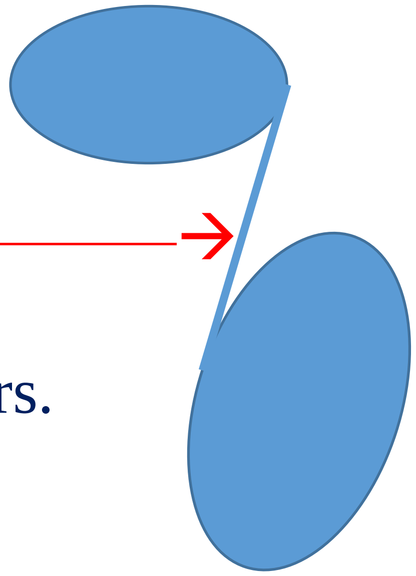
The Petersen graph

Contradiction !!!
So we MUST use 4 or more colors



The Danish mathematician, Julius Petersen (1839-1910), constructed the following graph, now bearing his name
— the *Petersen graph*
illustrating the smallest cubic graph with no isthmus that has edge-chromatic number greater than three.

- **cubic graph** — regular graph with vertex degree 3
- **no isthmus (bridge)** ———— *forbidden* ——→
- **edge-chromatic number** greater than 3
  — color the edges so touching edges have different colors.



Over the years, the Petersen graph and its generalizations have served as useful examples and counterexamples for many problems in graph theory.

William T. Tutte wrote in 1980: (translated from the original French)

At Cambridge University, I found a work of Sainte-Laguë entitled *Les Réseaux* (*ou Graphes*). There is a proof of Petersen's theorem.

I read. I understood. I filled the gaps.
I even made a small improvement in the result of the text.

'Look at you,' I said to myself,

'You can work on networks.
Perhaps the theory of graphs will be your research topic in the future!'

Petersen's Theorem. — *A cubic graph with fewer than three leaf components is reducible.*

Cubic graphs is the topic of Chapter V.

Dénes Kőnig cites *Les Réseaux* (*ou Graphes*) several times in his own 1936 book, *Theorie der endlichen und unendlichen Graphen*.

He mentions its *extensive bibliography* which was useful to him in investigating the development of early graph theory in the French mathematical literature.

ELSEVIER Discrete Mathematics 191 (1998) 91–99

On configurations and the book of Sainte-Laguë

Harald Gropp*

H. Gropp, Configurations and graphs, Discrete Math. 111 (1993) 269–276.

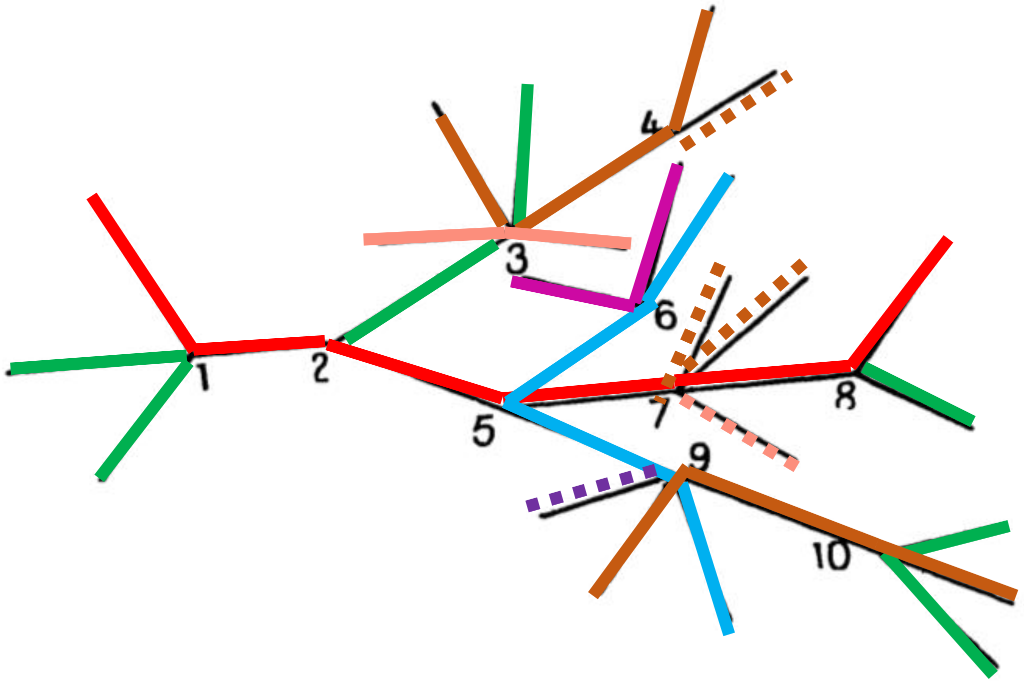
H. Gropp, On tactical configurations, regular bipartite graphs, and $(v, k$, even)-designs, Discrete Math. 155 (1996) 81–98.

H. Gropp, Poincaré and graph theory, Philosophia Scientiae 1 (4) (1996) 85–95.

H. Gropp, Configurations and graphs II, Discrete Math. 164 (1997) 155–163.

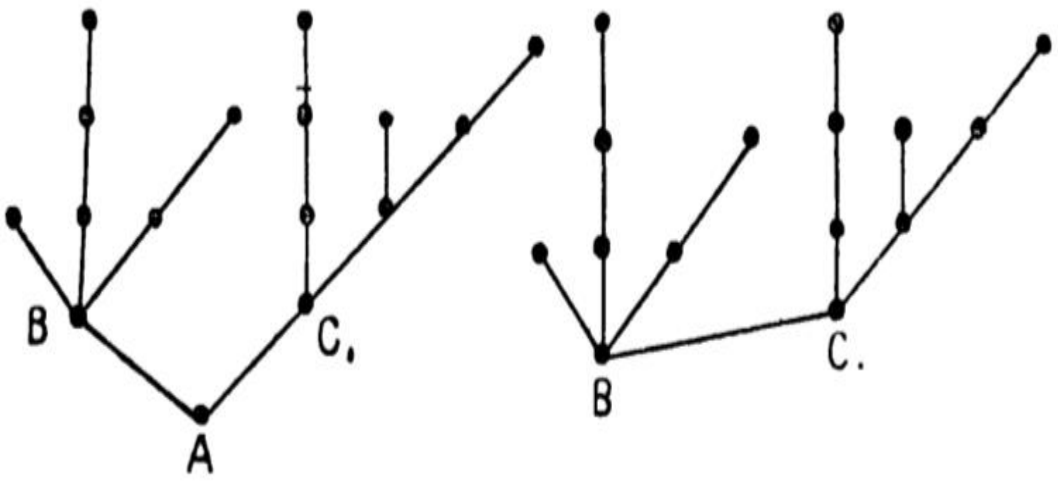
H. Gropp, Configurations and their realization, Discrete Math. 174 (1997) 137–151.

II. Trees



chain cover = trait (d'un arbre):
A subset of chains (or paths) such that each edge belongs to exactly one chain.

- A *tree* is a connected graph in which there is always a *unique chain of edges between any two vertices*.
- A *tree* is a connected graph *with no cycles*.



A typical counting result – In a tree T,

let m_2, m_3, … , m_p be the numbers of internal nodes of even degree: 4, 6, . . . , $2p$,
and n_2, n_3, … , n_q be the numbers of internal nodes of odd degree: 3, 5 , . . . , $2q + 1$.
The cardinality T of a chain cover of T is

$$T = 1 + \sum_{i\geq 2} i\, m_i + \sum_{j\geq 1} (j+1) n_j - \sum_{i\geq 2} m_i - \sum_{j\geq 1} n_j$$

- S.-L. includes dozens of such counting problems,
  mostly from cited papers and many of his own results.

As stated, the problem is ambiguous – *but that is not important !*

- It led researchers to define and study dozens of **new covering problems** on graphs.
- These raised hundreds of optimization variations and algorithmic questions
  enough for 10,000 Ph.D. students.

Let's mention just one of these variations from 2008.

Covering a Tree by a Forest

Fanica Gavril and Alon Itai

In Springer LNCS 5420, (M. Lipshteyn et al. , eds.), pp. 66–76, 2009.

Consider a tree T and a forest $\mathscr{F}$ (a collection of trees).

Forest vertex-cover problem (FVC): cover the vertices of T by a minimum number of copies of trees of $\mathscr{F}$, such that every vertex of T is covered exactly once.

Forest edge-cover problem (FEC): cover the edges of T by a minimum number of copies of trees of $\mathscr{F}$, such that every edge of T is covered exactly once.

Two versions of FVC are considered: *ordered covers* and *unordered covers*

Both have polynomial-time algorithms.

Two versions of FEC are considered: *ordered covers* and *unordered covers*

Both are NP-complete.

Another version of FEC, *consecutive covers*, is polynomial-time.

Trees Glorious Trees

Trees are pervasive – they are everywhere
-- the most important family of graphs.

In mathematics:

rooted and unrooted trees, bifurcating trees, random trees, multidimensional trees, Steiner trees, graceful trees and Catalan numbers to count trees.

In computer science:

binary/ternary/quad trees, splay trees, B trees, red-black trees, parse trees, AVL trees, electrical nets and a dozen species of search trees and spanning trees.

In operations research and economics: decision trees and game trees

In genealogy: matrilineal descent and patrilineal descent family trees

In biology: evolutionary trees and phylogenetic trees

In chemistry: molecular trees and Gutman trees, benzenoid trees.

Donald Knuth: "Trees sprout up just about everywhere in computer science."

André Sainte-Laguë

Sainte-Laguë was a pioneer of new educational technologies for teaching, promoting areas of recreational mathematics, and making mathematics accessible and understandable to the general public.

An actual classroom slide used by Sainte-Laguë to show the tree structure exhibited by sea corals.

Some problems on trees that have interested me

Let $\mathcal{P}$ be a set of paths $\{P_i\}$ on a tree T $(i = 1, \ldots, n)$.

We may consider several types of "*interaction*" between a pair of paths P_i and P_j

- P_i and P_j INTERSECT (*share at least one vertex of the tree*)

  Intersection graph of paths in a tree: $V(G) = \{1, \ldots, n\}$

  $v_i v_j \in E$ precisely when paths $P_i \cap P_j \neq \varnothing$ (share a vertex in T)

  These are called *path graphs* (Gavril, 1978) or *VPT graphs.*

- P_i and P_j EDGE INTERSECT (*share at least one edge of the tree*)

  Edge intersection graph of paths in a tree: $V(G) = \{1, \ldots, n\}$

  $v_i v_j \in E$ precisely when paths $|P_i \cap P_j| \leq 2$ (share an edge in T)

  These are called *EPT graphs* (Golumbic and Jamison, 1985)

- P_i and P_j k-EDGE INTERSECT (*share at least* k *edges of the tree*)

  $v_i v_j \in E$ precisely when paths $|P_i \cap P_j| \leq k + 1$ (share k edges in T)

  These are called *k-EPT graphs* (Golumbic, Lipshtein and Stern, 2005)

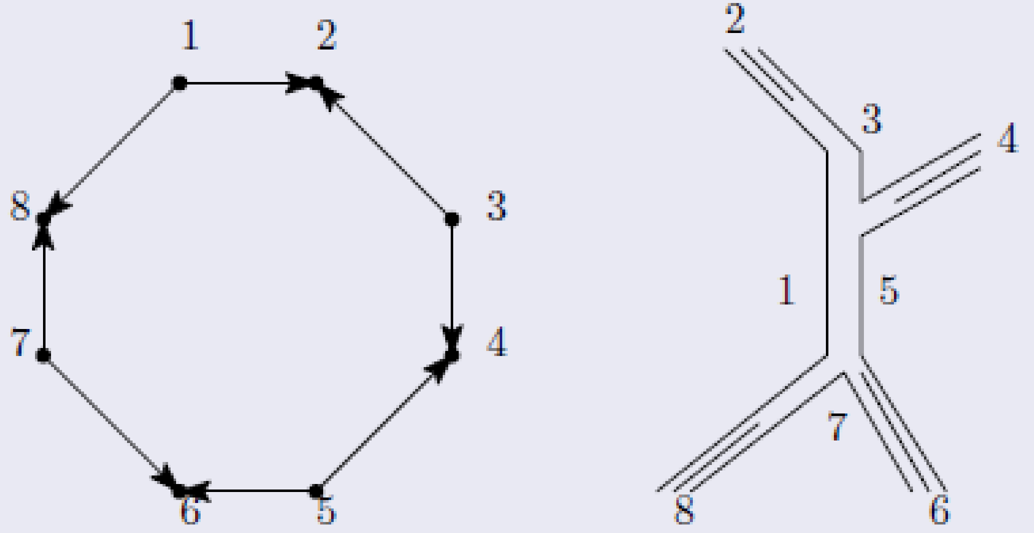
Containment graphs of Paths in a Tree (CPT)

Let $\mathcal{P}$ be a set of paths $\{P_v \mid v \in V(G)\}$ on a tree T.

CPT Graph: $vw \in E$ precisely when one of P_v or P_w contains the other.

CPT Order: $v \prec w$ precisely when $P_v \subset P_w$

Example C_8



Remark. The order $\prec$ gives a *transitive orientation* of G.

that is, if $a \rightarrow b$ and $b \rightarrow c$
then $a \rightarrow c$

Remark. Odd cycles have no *transitive orientation.*

The CPT property is NOT a Comparability Invariant

Definition: A property of an ordered set is a ***comparability invariant*** if **all** transitive orientations a given comparability graph have that property **or none** have that property.

The *poset dimension* is a well-known comparability invariant, that is,

dim(P) = dim(Q) whenever P and Q have the same comparability graph.

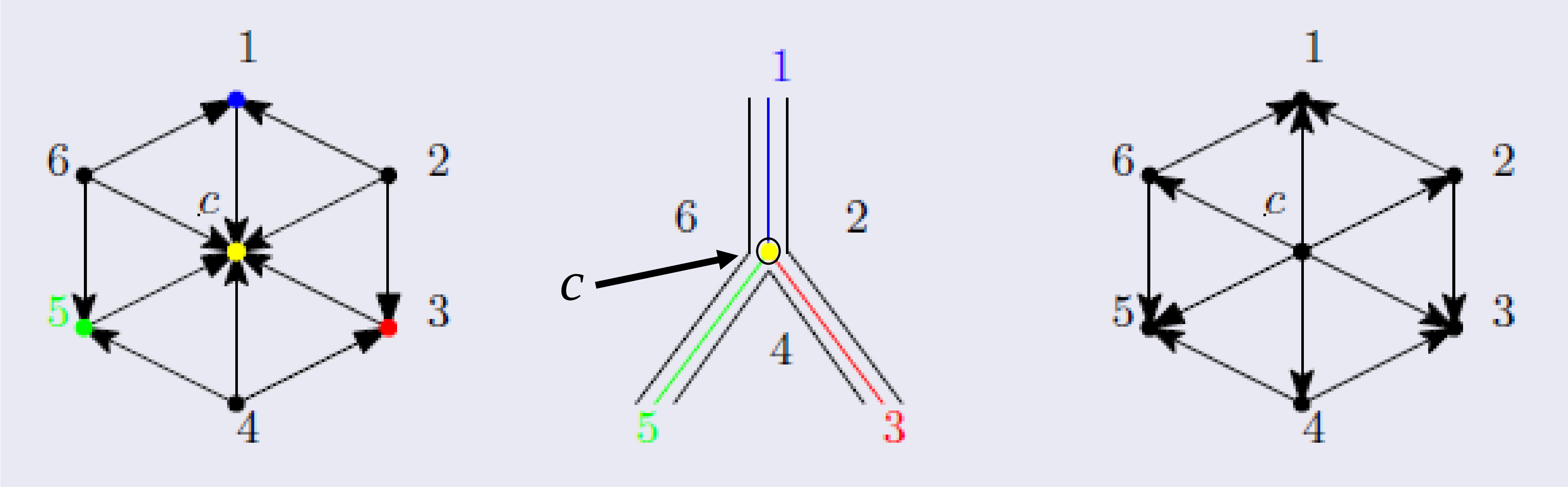
A proof of this result can be found in Chapter 7 of the book *Tolerance Graphs*.

Other invariant properties include:
interval order dimension, unit interval orders, box containment orders, semi-orders, jump number, bounded tolerance and bitolerance orders, unit tolerance, unit bitolerance orders and many more.

However, Corneil and Golumbic [1984], observed that being a CPT order is *not* a comparability invariant, as demonstrated by the wheel W_{2k} ($k \geq 3$).

The CPT property is NOT a Comparability Invariant

Example: W_6



In the 6-wheel W_6
its central vertex must be a sink in any CPT representation.

Proof. If c is a source, then its path P_c contains the paths P_1, P_2, P_3, P_4, P_5, P_6.
So they are intervals on the path P_c inducing C_6. This cannot happen since C_6 is not an interval containment graph.

Partial Wheels as CPT Graphs

Golumbic and Limouzy (*Order*, 2021),
Containment graphs and posets of paths in a tree: wheels and partial wheels.

Definition: A *partial wheel* is a wheel "missing some spokes", that is,
a chordless cycle and a central vertex adjacent to some but not all cycle vertices.

First Question:

(a) Which partial wheels have a transitive orientation?

(b) Which of those also admit a CPT representation?

First Result: *All partial wheels that admit a* TRO *are* CPT.

Two Goals:

Characterize the CPT orders whose comparability graph is a partial wheel.
Characterize the partial wheels for which every TRO is a CPT order.

Partial Wheels as CPT Graphs

Theorem A. Let W be a partial wheel. The following conditions are equivalent:

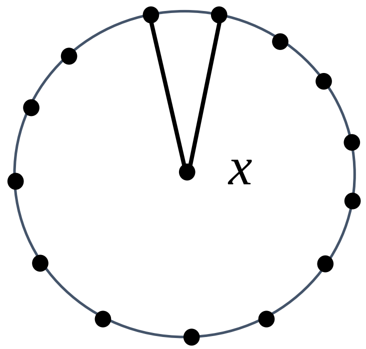
(1) W has a transitive orientation,

(2) W is a containment graphs of paths in a tree,

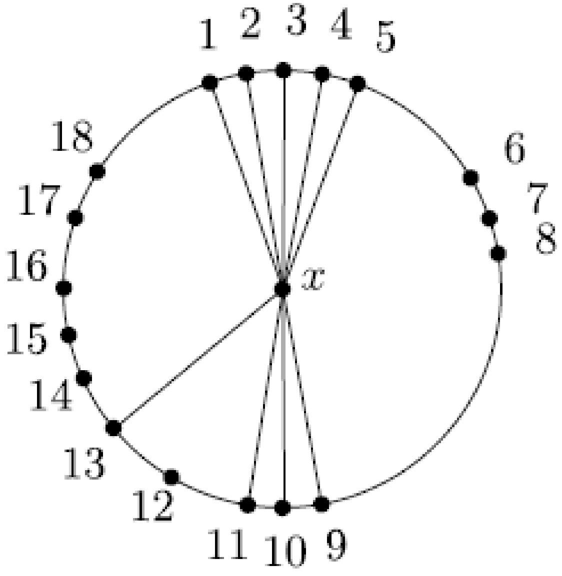
(3) the outer-cycle of W is of even length, and either

(a) the central vertex is adjacent to exactly two consecutive outer vertices, or

(b) all maximal sets of consecutive neighbors and of consecutive non-neighbors of the central vertex are of odd length.



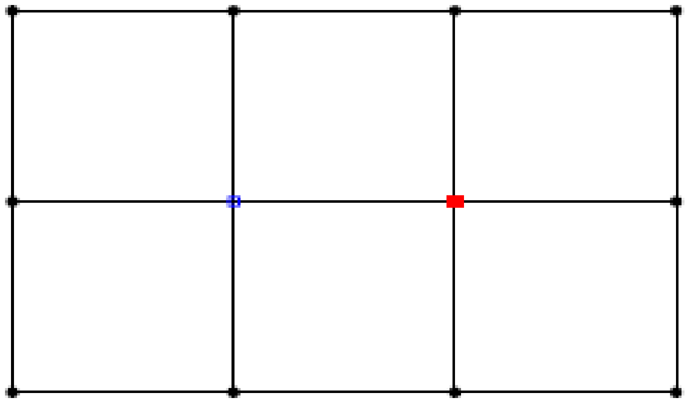
(a)

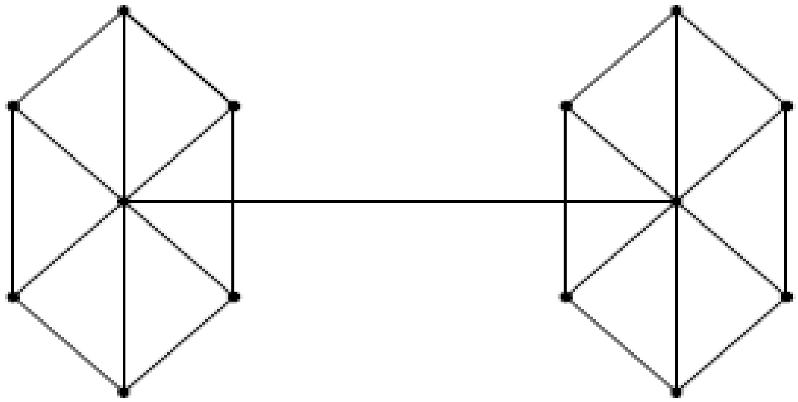


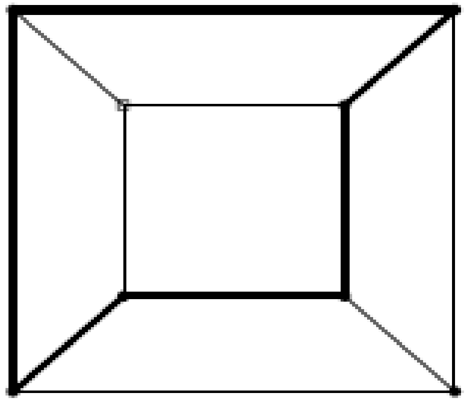
(b)

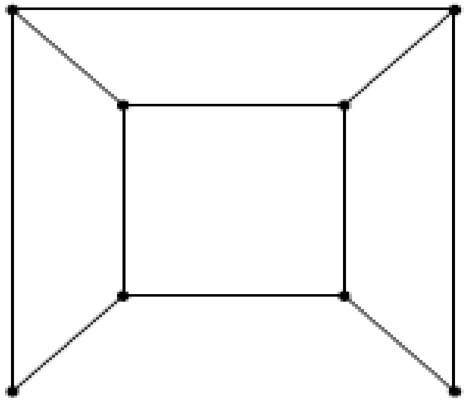
Some forbidden induced subgraphs for CPT graphs

consequences of our next result









Characterizing CPT orders

Theorem B. For wheels and partial wheels, the following characterizes their containment orders of paths in a tree.

(1) For the **full wheel** W_{2k} $(k \geq 3)$, the only transitive orientation which is CPT is that with the central vertex as a **sink**.

(2) For an even length partial wheel with the central vertex **adjacent to exactly 2** consecutive vertices, there are two transitive orientations and **both are CPT**.

(3) For an even length partial wheel with the central vertex **adjacent to exactly 3** consecutive vertices, there are four transitive orientations and **all are CPT**.

For any other partial wheel W of even length at least 6 satisfying condition (3)(b) of Theorem 1, we have the following:

(4) If the **gaps** of W are **all of length 1**, then the only transitive orientation which is CPT is that with the central vertex as a **sink**.

(5) **Otherwise**, there are two transitive orientations: with the central vertex as either a **sink or source**, and **both are CPT**.

Other CPT Results and Open Questions

1. Characterizing the CPT graphs and the CPT orders remain as open questions.
2. The same questions for bipartite CPT graphs.
3. For which CPT graphs will all transitive orientations admit CPT representations?
4. For which other comparability graphs will only one TRO be CPT and not its reversal, as in the case of full wheels?
5. Alcón, et al. (2018) characterized CPT split orders by a family of forbidden subposets. Similarly, questions of characterization and complexity can be asked about other subfamilies of CPT graphs and orders.
6. A CPT order P is called *dually*-CPT if both P and its dual P^d are CPT orders.

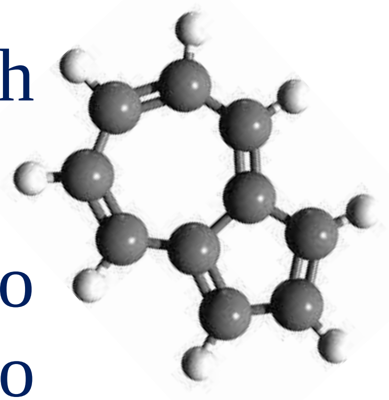
Characterize the *dually*-CPT orders. For example, in our Theorem B, statements (2) and (4) together characterize the dually-CPT orders of partial wheels.

Characterize other subfamilies.

Why are there so many variations of these problems?

Applications of Graph Theory

S.-L. (1926): "It seems that main applications of this branch of mathematics will be in Physical Chemistry or Organic Chemistry



-- the composition of matter and the *structure of crystals* seem to depend on graph theory, and the study of *paraffins* has given rise to interesting research."

It is "used in the study of many *games*, *invariants*, *determinants*, *analytic forms*, *arithmetic*, *groups of substitutions* and *permutations*."

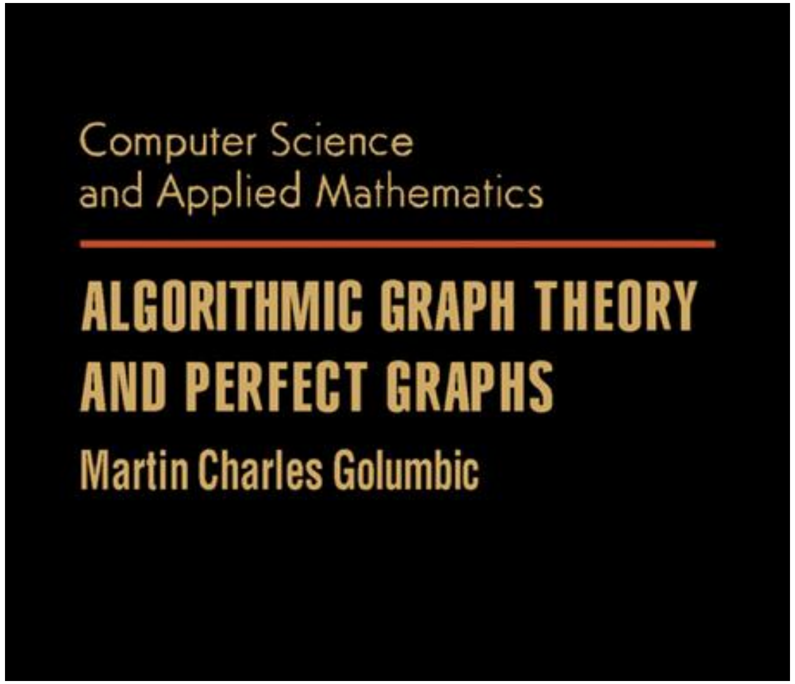
S.-L. does not mention *electric circuits*, even though an 1847 reference to Kirchhoff appears in his Bibliography.

Applications of Graph Theory

Marty (2021): But it is doubtful that anyone in 1926 with a passion for mathematics, science, or technology, could have imagined the *Voyages Extraordinaires* that graph theory has taken since.

This changed dramatically by 1958 when Claude Berge published his book *Théorie des graphes et ses applications*.

Then came computer science, operations research, and the algorithms revolution 1960-1980.



III. Chains and cycles

Eulerian chains and cycles

Does a given graph G admit a chain or cycle

passing through every edge exactly once?

S.-L. (1926) writes,

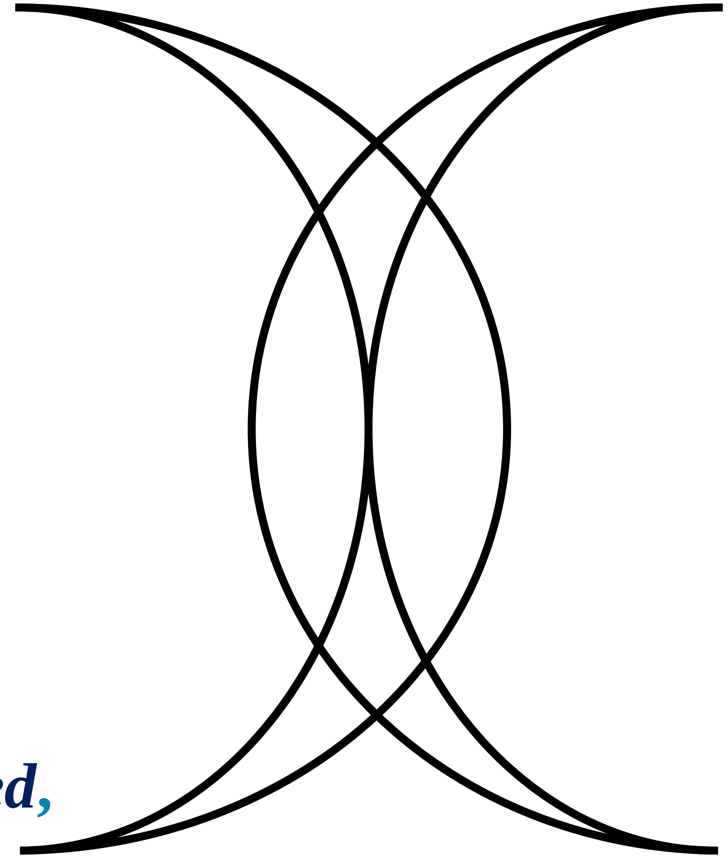
"This question was first asked by Euler,

(The Bridges of Königsberg Problem, 1736)

but must have been known before, as shown,

for instance, by the legend of the *Signature of Mohammed*,

which he traced with a tip of his sabre."



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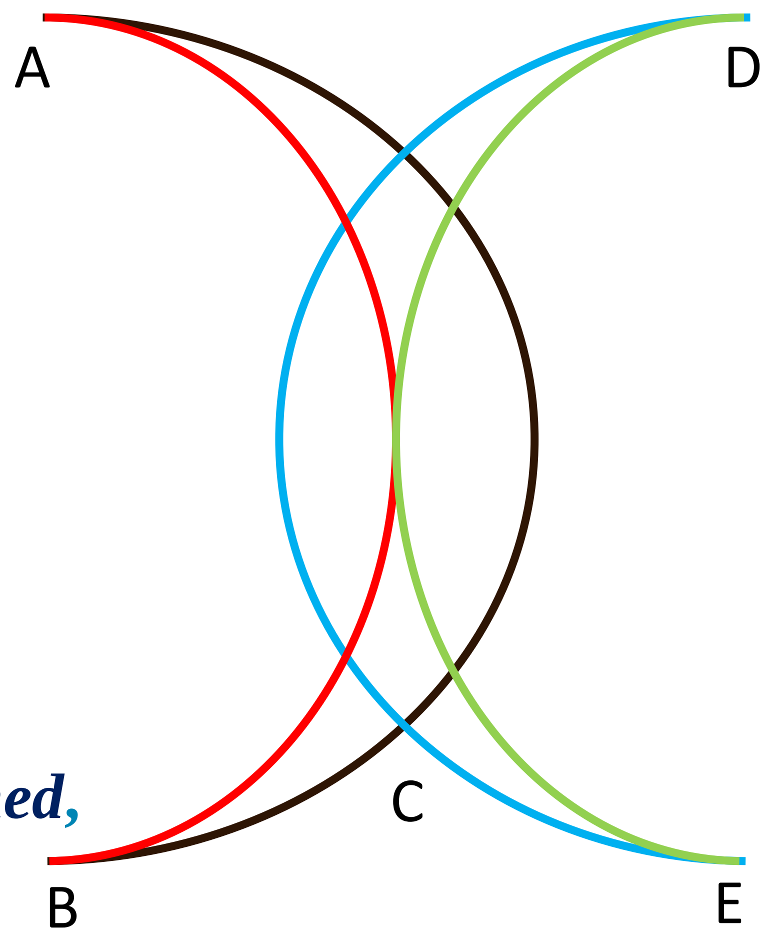

RED: A to B
BLACK: B to C
BLUE: C to D
GREEN: D to E
BLUE: E to C
BLACK: C to A

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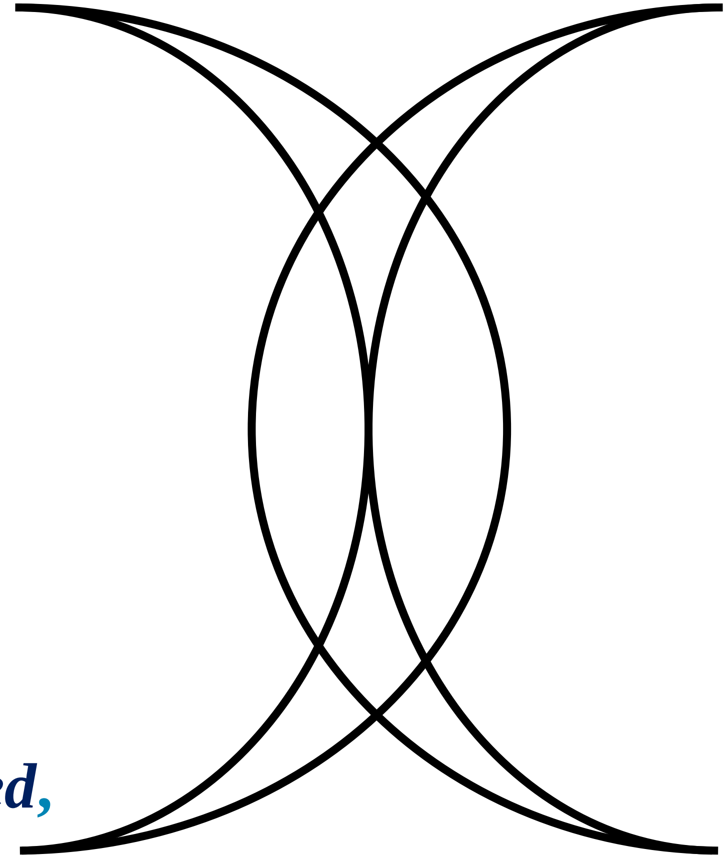


S.-L. (1926) writes, At the heart of such a study is the following obvious theorem:

The number of odd degree vertices of a graph is even.

WHAT's WRONG? He then correctly states:

Eulerian chains exist in a connected graph if and only if the number of odd vertices is 0 or 2.

S.-L. also writes, **“Fleury (1883) gave a practical procedure that allows us to find an ‘*entrelacement*’ (an Eulerian chain or cycle) in a given graph.”**

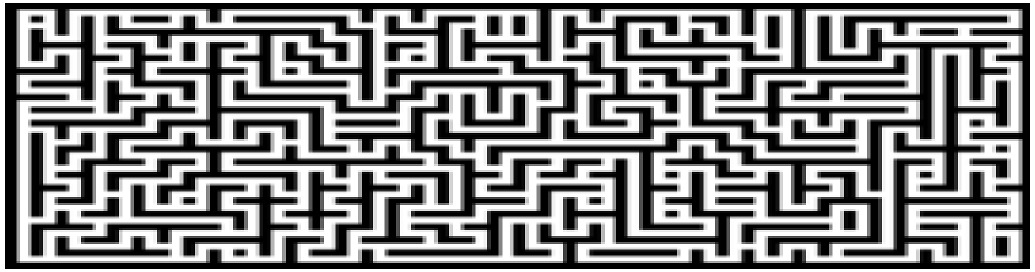
Commentary.

Fleury’s algorithm to compute Eulerian chains and cycles is still well known today.

Carl Hierholzer (1873) provides a different method for finding Eulerian chains and cycles.

Interestingly, *S.-L.* lists Hierholzer’s paper in his Bibliography, but never mentions the paper in the text.

Labyrinths



S.-L. writes,

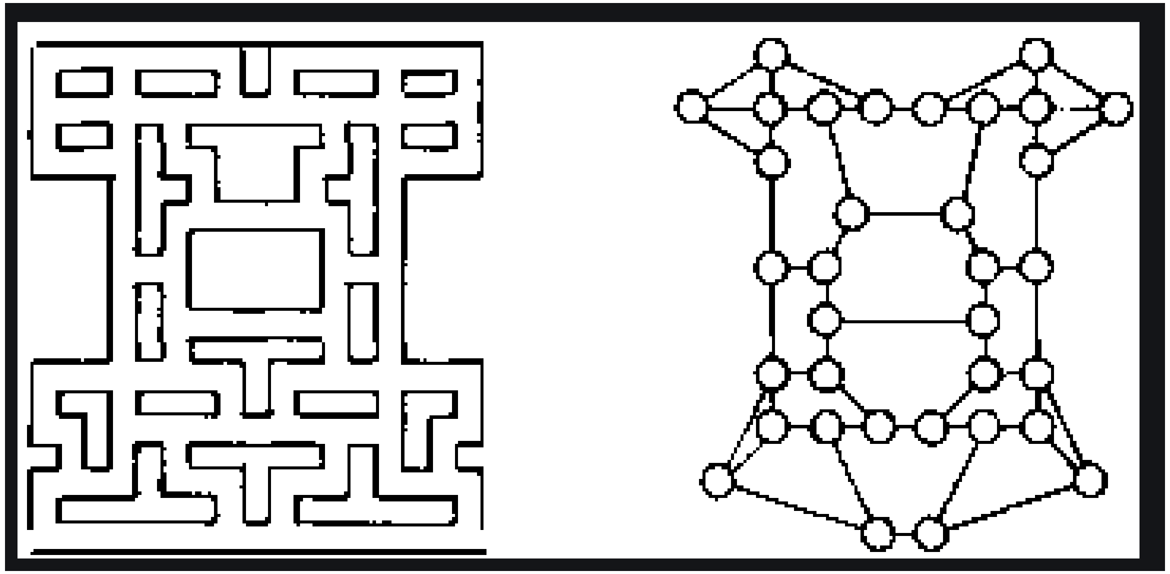
A *labyrinth* can obviously be represented by a graph.

Trémaux has shown that one can theoretically find a way out of a labyrinth —
that is, follow all edges in a graph, whose structure is unknown,
by applying a set of rules that he provided.

Charles Pierre Trémaux 1859-1882

Generally credited as being
the inventor of Depth-First Search (DFS).

Lucas refers to him as an ex-student
of the École Polytechnique
and a telegraph engineer.
He gives a full description of the method.



***Commentary*. Labyrinths, mazes and depth-first search**

- A very entertaining account of Trémaux's method appears in
  Édouard Lucas (1882), *The Game of Labyrinths* (Le jeu des labyrinthes)
- *S.-L.* seems not to have known about an earlier paper by
  Christian Wiener (1873).
- We should have expected *S.-L.* to cite two works by Tarry in his bibliography:
  Parcours d'un labyrinthe rentrant (1886); Le problème des labyrinthes (1895)
- English translations of these classical works, and the rigorous chapter
  The Labyrinth Problem from Kőnig's book

  can be found at Michael Behrend's website --

https://www.cantab.net/users/michael.behrend/repubs/maze_maths/pages/index.html

The number of Eulerian cycles in a graph

S.-L.: "Delannoy, Tarry and Lucas have calculated the number of distinct Eulerian cycles in a given graph."

He then presents many such counting results for various types of graphs.

Let us remark (Métrod, 1917) that the number of Eulerian chains in a graph with 3 vertices connected pairwise by *a*, *b*, and *c* edges, respectively, either all even or all odd, is

$$2b!c!\left(\frac{a+b}{2}-1\right)!\left(\frac{a+c}{2}-1\right)!\sum_k \frac{(a+k-1)!}{k!\left[\left(\frac{a-k}{2}-1\right)!\right]^2\left(\frac{b-k}{2}\right)!\left(\frac{c-k}{2}\right)!}$$

where *k* in the sum takes values of the same parity as *a*, *b*, and *c*,

taking successively the values up to the smaller of the numbers *b* and *c*.

You might ask,

Marty, Why did you write this book, and what do you think people will get out of reading the book?

VI. Tableaux

Incidence Matrices and Semi-regular matrices (*Tissus*)

tissu — the term used by *S.-L.* for a *binary matrix with constant row and column sums.*

In French, *tissu* means fabric or cloth. WHAT does this have to do with a matrix?

The matrix might remind one of woven textiles.

Lucas introduced the topic of *géométrie du tissage* in his papers 1867-80

inspired by principles of weaving fabric with rectilinear threads.

The term "semi-regular matrix" was introduced by Brualdi (1980).

the number of 1s in each row is constant,
and
the number of 1s in each column is constant

Commentary. The *geometry of fabrics* has become a well-studied area, branching off into many mathematical directions of research. See the papers of Grünbaum and Shephard, Beauville, and the extensive 1999 bibliography by Joseph Malkevitch.

Commentary. *Tissus, mosaïques, et échiquiers*

Anne-Marie Décaillot (2002), *The geometry of fabrics, mosaics, chessboards: Curious and useful mathematics*

In the second half of the nineteenth century, a group of mathematicians,
driven by a common ambition of disseminating science to a wide audience,
began to treat mathematical questions originating in concrete problems.

One of their favorite techniques was to employ the well-known common chessboard.
It suggested Lucas's *geometry of fabrics* with connections to number-theory.
Then came Laisant's construction of mosaics related to finite groups and crystallography.
James Joseph Sylvester's analagmatic chessboards represented examples of recreational mathematics before their transformation into matrices attracted Jacques Hadamard.

In this same spirit, André Sainte-Laguë was influenced by the work of these nineteenth century mathematicians. **He continued carrying this message of making mathematics popular throughout his career.**

VII. Hamiltonian graphs -- *Réseaux cerclés*, in the terminology of S.-L.

A graph is *Hamiltonian* (*cerclé*, in the terminology of *S.-L.*)
if it admits a cycle that goes through all the vertices of the graph.

S.-L. writes,

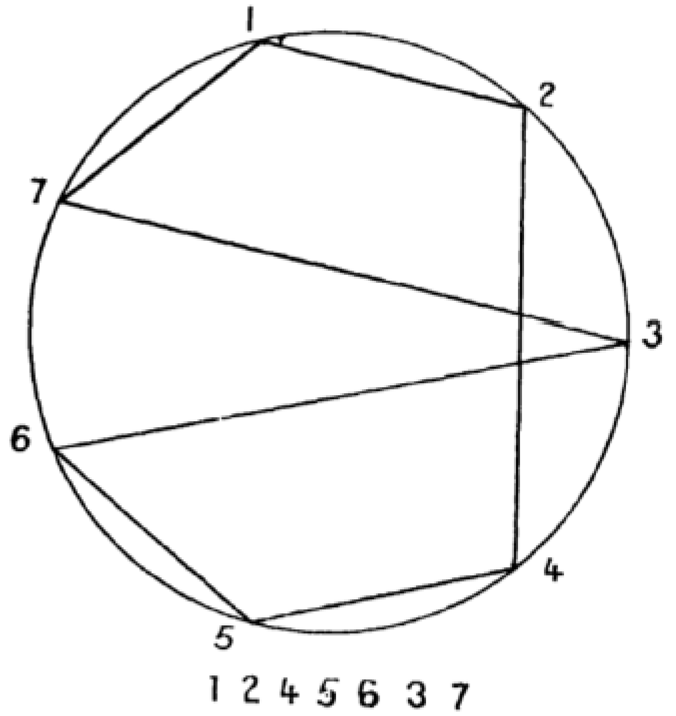
For small values of *n*, bipartite cubic graphs are Hamiltonian, but this is not true for any *n*,
moreover, *every graph with an isthmus is not Hamiltonian.*

If we examine the simplest regular graphs, we find that
for degree 2, they are all Hamiltonian.

The degree 3 graphs are Hamiltonian graphs for $n = 4, 6, 8$;
for $n = 10$, out of 19 graphs, two are not Hamiltonian, of which one has an isthmus;
for $n = 12$, out of 80 graphs, five are not Hamiltonian, of which four have an isthmus.

For degree 4, $n = 5, 6, 7, 8, 9$ give 16 graphs, all Hamiltonian,
and $n = 10$ gives 57, of which two are not Hamiltonian.

The *graph of the permutation*, a regular Hamiltonian graph, of order *n*.



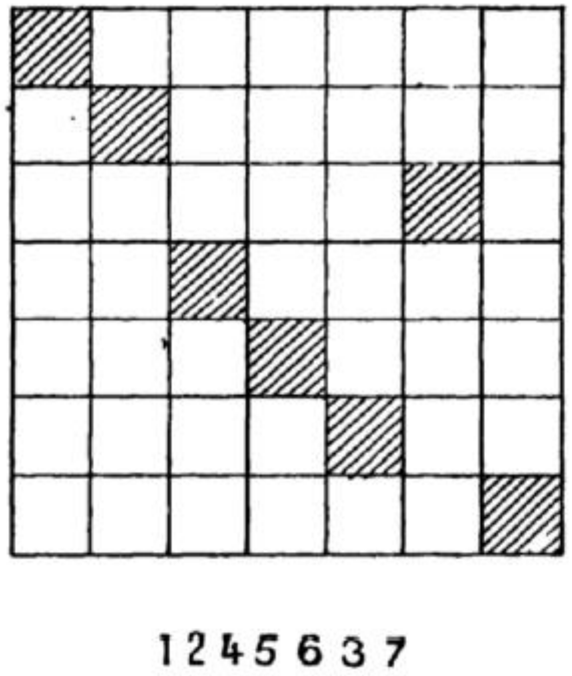


Fig. 10. The table for the permutation *A*

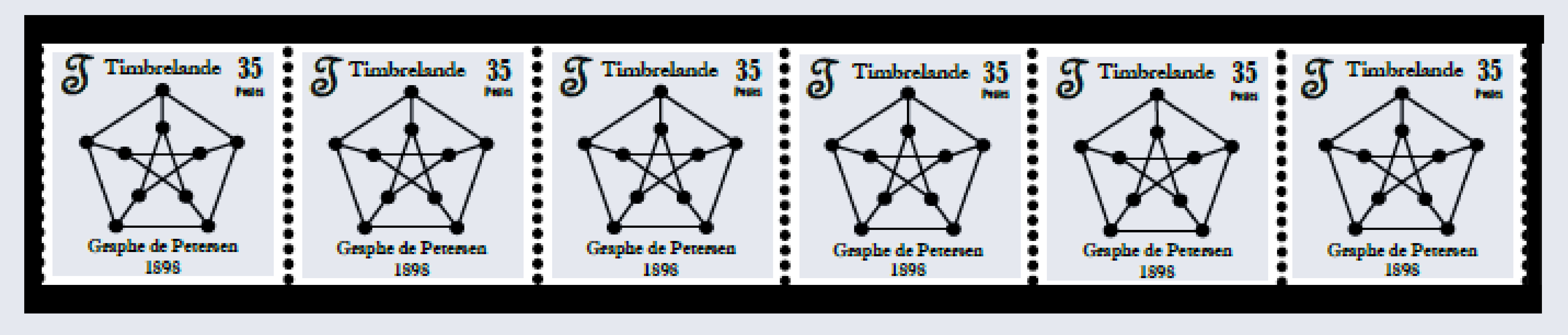
Circular permutations, additive permutations, *inverse permutations*, *complementary permutations*, *reciprocal permutations*

The problem of postage stamps. — In Lucas [1891] we find the following question:

In how many ways can one fold a strip of postage stamps?

Despite its apparent simplicity, this question remains unsolved.
If we decide, before folding, to number the stamps 1, 2, …, *n*, then the folded strip will give, from top to bottom, a certain permutation. We have to distinguish *permutations that can arise this way from those that cannot.*

Update in: Legendre S. (2014). Foldings and meanders, *Australasian Journal of Combinatorics* 58 (2) 275-291.



Stamps designed by the artist Edmond Henri Becker (20.07.1871 – 02.11.1971)

The 1924 Olympics were the first to use

- the official *marathon distance* of 42.195 km (26.219 miles),

  fixed by the International Amateur Athletic Federation (IAAF) in May 1921, and

- the standard 50m pool with marked lanes.
- During the games, British runners Harold Abrahams and Eric Liddell won the 100m and 400m events, respectively. Their stories are depicted in the 1981 movie *Chariots of Fire* whose title is said to be inspired by the line, “Bring me my chariot of fire!”, from a William Blake poem, and the original Biblical phrase רֶכֶב אֵשׁ in II Kings 6:17.

VIII. Chessboard problems

Inspired by Ahrens, Lucas and Rouse-Ball and the "Anallagmatic" chessboards of Sylvester (1868).

The term "***anallagmatic***", from the Greek for "unchanging", refers to an object or structure that is not changed in form by inversion.

Sainte-Laguë: The points of the plane with integer coordinates form a graph that we can consider as an infinite chessboard (*échiquier indéfini*).

- The number of moves of a chess piece, and pairwise non-attacking rooks, queens, etc.
- Great queens, half-queens, half-bishops and 1000 other unorthodox, unconventional chess pieces from the Middle Ages.
- Magic squares and Knight's tours.

Commentary. An infinite chessboard from medieval Portugal.

Image: common street tiles, 2020 – © Martin Charles Golumbic

Problems on Queens

Is it possible to place 16 queens on a chessboard of 64 cells so that, on each row, column, or parallel to a diagonal, a queen attacks at most one other queen?

This question assumes that there are two queens per row and per column, and leads to double permutations such as the following one, which gives a solution to the problem [Ahrens, 156].

4 3 1 2 1 2 4 5
8 5 7 3 7 6 6 8

			Q		Q		
		Q		Q			
				Q		Q	
	Q						Q
Q						Q	
	Q	Q					
			Q		Q		
Q							Q

Commentary. An illustration of Sainte-Laguë's example of 16 queens.

Another Problem on Queens

On a chessboard of 49 cells [156], can we place 49 queens of 7 different colors so that no two queens of the same color attack each other.

Yes: Subsequent rows are deduced by the circular permutations indicated by the colors of the first column:

A; C; E ;G; B; D; F.

Commentary.

A	B	C	D	E	F	G
C	D	E	F	G	A	B
E	F	G	A	B	C	D
G	A	B	C	D	E	F
B	C	D	E	F	G	A
D	E	F	G	A	B	C
F	G	A	B	C	D	E

Problems on Knights

1. How many knights do we need to place on a chessboard in order to attack all cells?

2. ***Knight's tour problem:*** How can we find all the chains [sets of moves] of a knight across a chessboard, so that it visits each cell once and only once?

Robin Wilson comments: The problem is **over 800 years old**, and was solved pre-Euler. Euler was the first to study it mathematically..

Algorithmically, the knight's tour problem can be solved in linear time, unlike the general Hamiltonian problem which is NP-complete.

3. How many knight's tours are there?

The classical solution of Euler:

13	54	21	64	15	62	23	58
20	51	14	55	22	59	16	61
53	12	49	18	63	10	57	24
50	19	52	11	56	17	60	9
5	48	27	40	7	36	25	34
28	14	6	45	26	33	8	37
47	4	43	30	39	2	35	32
42	29	46	3	44	31	38	1

“Modern Methods”

Sainte-Laguë in 1926:

Section 82. **Modern methods.**— An ingenious solution to the *problem of Euler* was given by Warnsdorff ….

Another interesting method is one of Roget.

Commentary. Peter Mark Roget

Peter Mark Roget, an English physician, is remembered mostly for his *Thesaurus of English Words and Phrases* (1852), but he also invented a “log-log” slide rule for calculating the roots and powers of numbers. He was an avid chess player and solved the general open knight’s tour problem [218]. See the article, *Peter Mark Roget and Chess*, by Edward Winter at https://www.chesshistory.com/winter/extra/roget.html.

Knight’s tour magic square

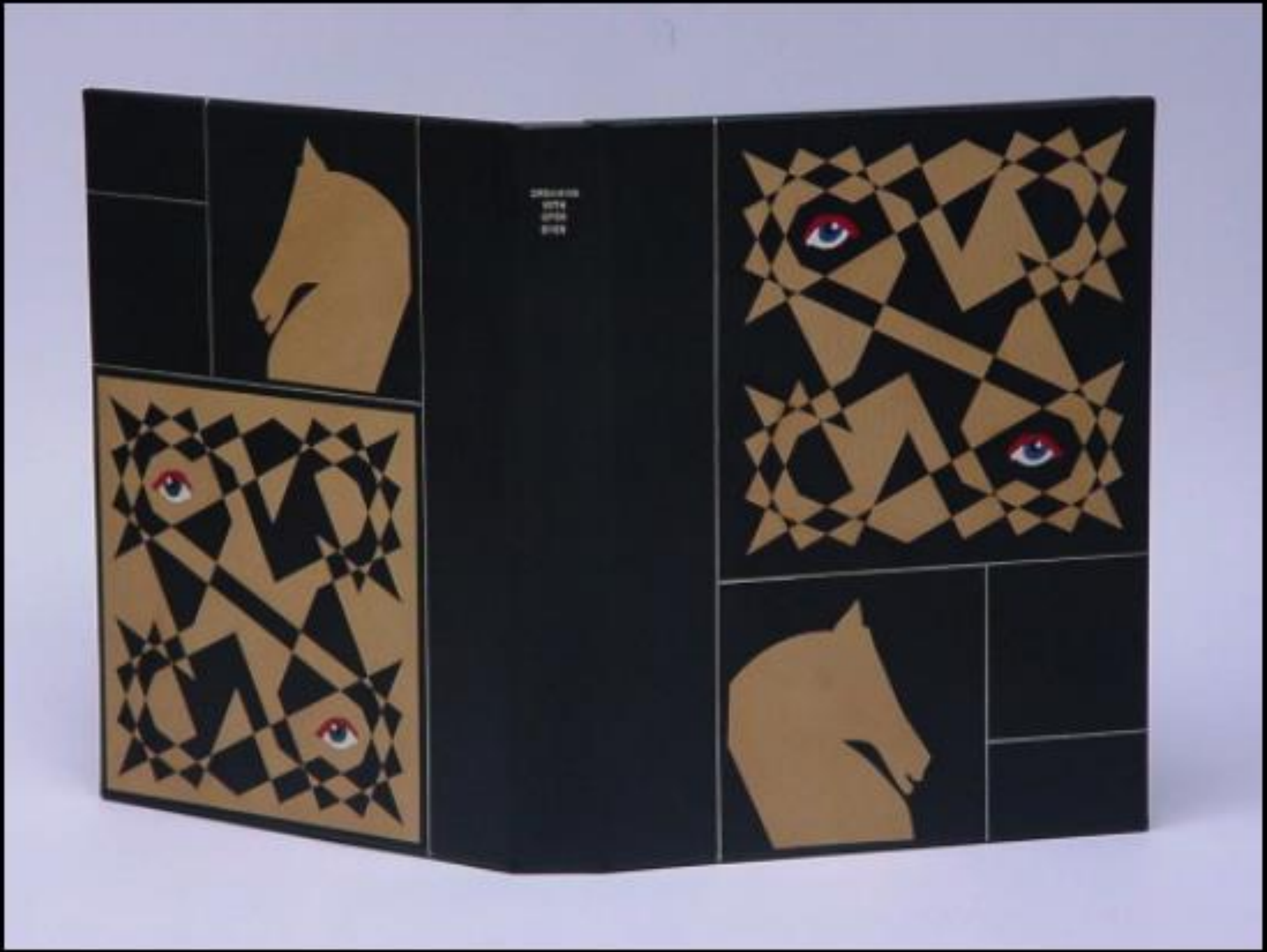
Numbering the cells in the order passed through by the knight obtaining a *magic square*: the sum in each row or column, *but not a diagonal*, is constant.

Example: Constant equal to 260.

47	10	23	64	49	2	59	6
22	63	48	9	60	5	50	3
11	46	61	24	1	52	7	58
62	21	12	45	8	57	4	51
19	36	25	40	13	44	53	30
26	39	20	33	56	29	14	43
35	18	37	28	41	16	31	54
38	27	34	17	32	55	42	15

- Investigated first by William Beverley (1848), then Carl Wenzelides (1849), Krishnaraj Wadiar (1850s), C. F. de Jaenisch (1862) and E. Francony (1882).
- Extensive histories of *knight's magic tours* are given by Murray (1951) and by George Jelliss (2002) at https://www.mayhematics.com/t/1h.htm
- There are a total of **140 distinct magic knight's tours** on the 8 x 8 board -- completed by an exhaustive computer enumeration, software written by J. C. Meyrignac (August 2003), see http://magictour.free.fr/

A gift box for Arturo Schwartz.
Full green goatskin with inlaid design based on the magic square of eight (the knight's tour).



Yehuda Miklaf

- Jerusalem artist and Bookbinder

His first commission as a professional binder was for a book of photographs by Tom Moore, presented to Queen Elizabeth II by the government of Ontario for her silver jubilee in 1977.

There Are No Magic Knight's Tours on the Chessboard

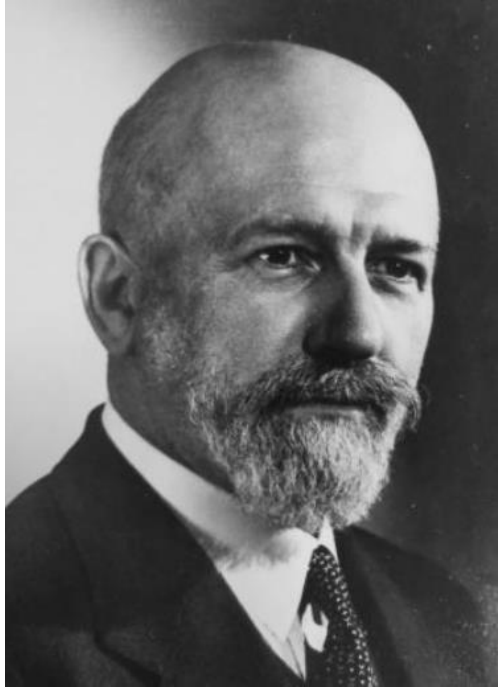
***MathWorld* Headline News** August 6, 2003, by Eric W. Weisstein

What? ! ! !

There is no 8 x 8 **FULLY diagonal** magic knight's tour possible:
that is, the diagonals must also sum to the magic constant.

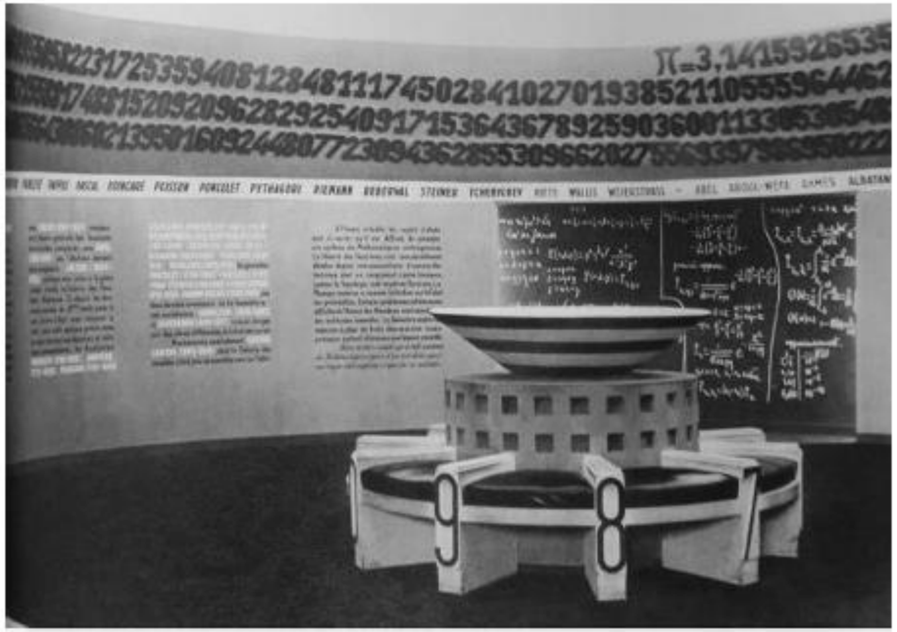
A 150-year-old unsolved problem – finally has been answered by software written by J. C. Meyrignac, see the website http://magictour.free.fr after 61.40 CPU-days, corresponding to 138.25 days of computation at 1 GHz,

André Sainte-Laguë in 1937



Sainte-Laguë was entrusted by Émile Borel with the organization of the mathematics rooms in the Palais de la Découverte (Palace of Discovery), the science museum created for the 1937 Paris Exposition.

You can visit it today at the Grand Palais in Paris, including the famous **Pi Room.**



The Pi Room in 1937

The Pi Room in 2020

Also a film, *De la similitude des longueurs et des vitesses* (Similarities between length and speed) was written by André Sainte-Laguë and directed by Jean Painlevé.

Historical Note: Jean Painlevé

A filmmaker, well-known for his scientific documentaries

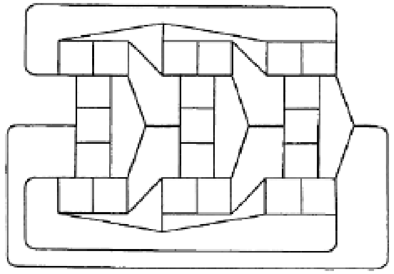
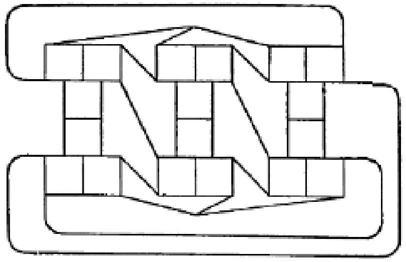
https://jeanpainleve.org/

Son of Paul Painlevé, a mathematician and twice prime-minister of France, who appointed Émile Borel as Minister of Marine during his second (7 month) prime-ministry in 1925.

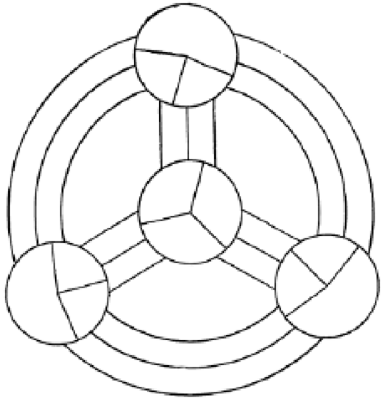
Sainte-Laguë published several works with his colleague Antoine Magnan, on the aerodynamics and flights of birds, gliders, and planes, and an essay on the motion of fish in the 1930s.

His 1937 book, *Avec des Nombres et des Lignes: Récréations Mathématiques,* was reprinted in 1994 and 2001, edited by André Deledicq and Claude Berge.

He also wrote about the symmetries of nature, the world of form, and authored the book, *From Man to Robot*, 1953.

Geometric figures from Sainte-Laguë (1929)
Géométrie de Situation et Jeux



Conclusion S.-L.:

The study of graphs can be pursued in many different ways, and each of the notions defined may initiate new research.

We have investigated, to the best of our ability, the complexity of issues that are raised, and the variety of methods that have been employed and are indispensable.

The subject, as limited as it may appear at first, is in fact vast and seems quite difficult. The research of Lucas and Tarry relate immediately to the theory of networks and graphs.

Other applications of more immediate utility could also be considered. There is further work already started on planar graphs, and practical applications of graphs, as well as many questions involving higher arithmetic, topology, and game theory.

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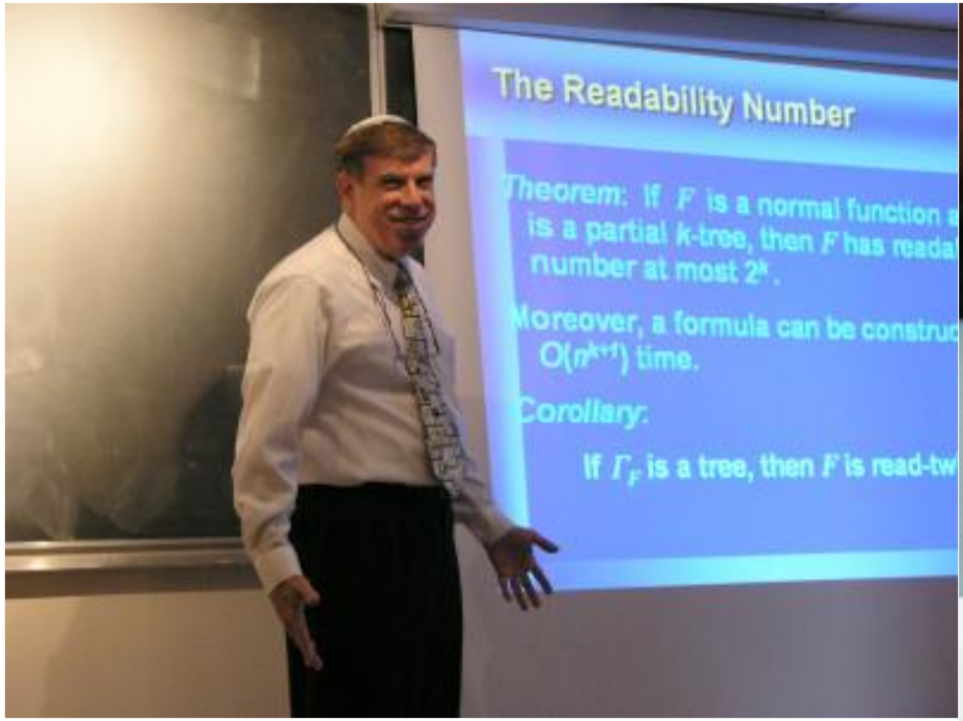
Merci Paris



LIAFA 2005
visiting
Michel Habib