Graph Encoding of 2D-gon Tilings

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Abstract: $2D$-gons tilings with parallelograms are the main model used in physics to study quasicrystals, and they are also important in combinatorics for the study of aperiodic structures. The algorithmic manipulation of these objects is not easy because of their geometrical nature. In this paper, we present an encoding of $2D$-gons tilings with graphs based on the adjacency relation between tiles. We show that the graphs we introduce are in one-to-one correspondence with the considered tilings, which make it possible to use them as dual objects. This encoding can be used to make various statistical studies and experiments on these tilings. One can for example use it to sample random $2D$-gon tilings.

Important note: This *extended abstract* has been prepared for submission to the ICALP 2002 conference. The proofs will be published in a journal version of the paper. The extended abstract comes with an appendix showing how our results can be used to sample random tilings. This appendix is not aimed to be published in the proceedings of ICALP; it has been added to help the evaluation of the paper by the referees.

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Abstract: 2D-gons tilings with parallelograms are the main model used in physics to study quasicrystals, and they are also important in combinatorics for the study of aperiodic structures. The algorithmic manipulation of these objects is not easy because of their geometrical nature. In this paper, we present an encoding of 2D-gons tilings with graphs based on the adjacency relation between tiles. We show that the graphs we introduce are in one-to-one correspondence with the considered tilings, which make it possible to use them as dual objects. This encoding can be used to make various statistical studies and experiments on these tilings. One can for example use it to sample random 2D-gon tilings.

1 Introduction

A tiling can be defined as a partition of a given region of an affine space. More classically, one considers a finite set of shapes, called *prototiles*, and a region of an affine space. The tiling problem is then to decide whether this region can be tiled, i.e. covered by translated copies of prototiles, without gaps or overlappings between them. If this is possible, the region is *tilable*, and a solution is called a *tiling* of the region. The translated copies of the prototiles are the *tiles* of the tiling. If the region to tile is the whole plane, this problem has been shown to be undecidable by Berger [Ber66], which was the first important incursion of tilings in computer science.

In this paper, we are concerned with tilings of $2D$-gons with parallelograms. A $2D$-gon is a convex finite region of the affine plane: an hexagon when D=3, an octagon when D=4, a decagon when D=5, etc. Such a region can always be tiled with parallelograms. $2D$-gon tilings by parallelograms appear in physics as a model for quasicrystals [Des97] and aperiodic structures [Sen95]. They are also used to encode several combinatorial problems [Eln97, Lat00], and have been studied from many points of view [Lat00, RGZ94, Bai99]. In particular, they are strongly related to the matroid theory, since the Bohne-Dress theorem proves the equivalence of $2D$-gon tilings with a class of matroids [RGZ94, BVS$^+$99].

These objects cannot be easily manipulated by a program when one uses the previous geometric definitions. That is why a notion of dual grid, the *de Bruijn grid* [dB81], has been introduced. However, this grid still contains geometric information. We will see in the next section that if we consider this grid as a graph (the *adjacency graph*), then some tilings which we want to distinguish give the same graph. The aim

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of this paper is to provide an effective notion of dual graph of a tiling. We will first present the tilings more formally, and define the adjacency graph we will use. Then we will obtain a one-to-one correspondence between a class of graphs and $2D$-gon tilings by introducing the notion of *graph with origins*, and we give an algorithm which builds the $2D$-gon tiling corresponding to a given graph with origins.

2 Preliminaries

Given two vectors v and v', we will say that $v < v'$ in the *natural order* if the angle between $(1,0)$ and v_i is smaller than the angle between $(1,0)$ and v_{i+1}. Let $V = \{v_1, v_2, ..., v_D\}$ be a family of D pairwise non-colinear positive vectors of the plane. We suppose that for all integer i, $v_i < v_{i+1}$ in the natural order. Let $M = \{m_1, m_2, ..., m_D\}$ be a family of D positive integers. The integer m_i is called the *multiplicity* of v_i. The $2D$-gon $P = (V, M)$ associated to V and M is the region of the affine plane defined by:

$$\left\{ \sum_{i=1}^{D} \lambda_i v_i, 0 \le \lambda_i \le m_i, m_i \in M, v_i \in V \right\}$$

There exists many equivalent definitions for these objects. For example, a $2D$-gon can be viewed as the projection of a hypercube of dimension D onto the plane. See [Zie95] for more details. For D=2, the $2D$-gons are parallelograms; for D=3, hexagons; for D=4, octagons; for D=5, decagons; etc. See Figure 1 for an illustration.

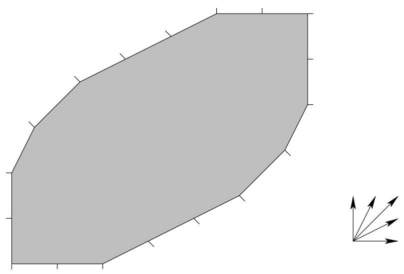

Figure 1: A $2D$-gon and its vectors, associated to $M = \{2, 1, 1, 3, 2\}$. This is a decagon with sides of length 2,1,1,3,2,2,1,1,3,2.

Given a $2D$-gon $P = (V, M)$, a prototile of P is a $2D$-gon built using only 2 vectors in V, each of them with multiplicity 1. Therefore, each tile of P is a parallelogram defined by two vectors in V, and we will make no distinction between the prototile viewed as an area and the pair of the indexes of the vectors in V which define it.

Finally, a tiling T of a $2D$-gon $P = (V, M)$ is a set of tiles (i.e. translated copies of the prototiles) which cover exactly P and such that there is no overlapping between tiles. Therefore, T is a set of couples, their first component being the pair of vectors which defines the tile, the second one being a translation. The translations used in $2D$-gon tilings can always be written as a linear combination of vectors in

$V : t = \sum_i \alpha_i v_i$, α_i being an integer between 0 and m_i. Two tilings T and T' of two $2D$-gons P and P' are said to be *equivalent* if $T = T'$, where T and T' are viewed as sets of couples.

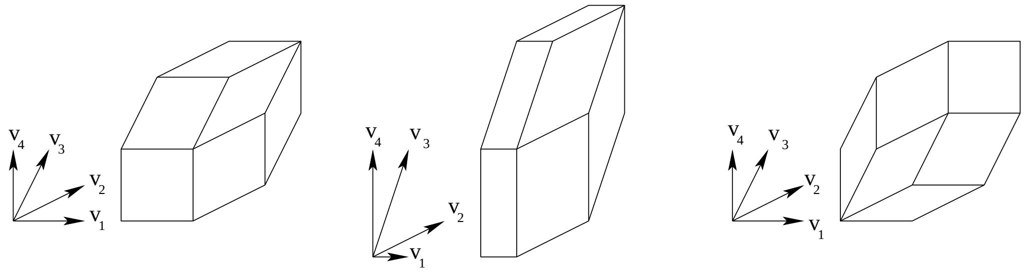

Figure 2: Three tilings of $2D$-gons, namely T_1, T_2 and T_3; T_1 and T_2 are *equivalent*, whereas T_3 is equivalent to none of the others.

Consider for example the three tilings of Figure 2. From left to right, they are described by :

$$T_1 = \left\{ \begin{array}{l} (\{1,4\},(0,0,0,0)),(\{2,4\},(1,0,0,0)),(\{3,4\},(1,1,0,0)),(\{2,3\},(1,0,0,1)), \\ (\{1,3\},(0,0,0,1)),(\{1,2\},(0,0,1,1)) \end{array} \right\}$$
$$T_2 = \left\{ \begin{array}{l} (\{1,2\},(0,0,1,1)),(\{1,3\},(0,0,0,1)),(\{1,4\},(0,0,0,0)),(\{2,3\},(1,0,0,1)), \\ (\{2,4\},(1,0,0,0)),(\{3,4\},(1,1,0,0)) \end{array} \right\}$$
$$T_3 = \left\{ \begin{array}{l} (\{1,2\},(0,0,0,0)),(\{1,3\},(0,1,0,0)),(\{1,4\},(0,1,1,0)),(\{2,3\},(0,0,0,0)), \\ (\{2,4\},(0,0,1,0)),(\{3,4\},(0,0,0,0)) \end{array} \right\}$$

Therefore, tilings T_1 and T_2 are equivalent, while T_1 and T_3 are not. Let P be a $2D$-gon, and T be a tiling of P. The i-th *de Bruijn family* of T is the set of all the tiles in T which are built with the vector v_i. Moreover, each family can be decomposed into *de Bruijn lines*: the j-th de Bruijn line of the i-th family is the set of tiles built with v_i which have $j-1$ as the i-th component of their translation. Continuing with our example of Figure 2, we obtain that the first line of the second de Bruijn family is equal to $\{(\{2,4\},(1,0,0,0)),(\{2,3\},(1,0,0,1)),((2,1),(1,0,1,1))\}$. For practical convenience, we will also say that the j-th line of the i-th family is the α-th line of the tiling where $\alpha = \sum_{k=1}^{i-1} m_k + j$, m_k being the multiplicity of the k-th family. We define $f(i)$ as the index of the vector associated to the i-th line, i.e. $f(i)$ is the number of the line's family. See Figure 3 for another example. Notice that two lines in the same family never have a tile in common, whereas two lines in two different families always have exactly one tile in common. We will use this fondamental property in the following.

Before entering in the core of the paper, we need a few more notations, which we introduce now.

Definition 1 *Let $P = (V, M)$ be a $2D$-gon. For all k, $1 \leq k \leq D$ we define the k-th* side *of P as the set of points:*

$$\left\{ \sum_{i=1}^{D} \lambda_i v_i, \forall i < k; \lambda_i = m_i, \forall i > k : \lambda_i = 0, \textit{and } 0 \leq \lambda_k k \leq m_i \right\}$$

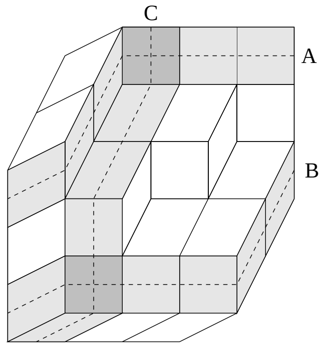

Figure 3: A $2D$-gon tiling and three de Bruijn lines (a line is a set of tiles crossed by a dotted line). A and B are in the same de Bruijn family. A is the third line of the third family, and B the second line of the same family. C is the first line of the first family.

Likewise, we define the $(k + D)$-th side of P as:

$$\left\{ \sum_{i=1}^{D} \lambda_i v_i, \forall i > k; \lambda_i = m_i, \forall i < k : \lambda_i = 0, \textit{and } 0 \leq \lambda_k k \leq m_i \right\}$$

Moreover, the hull *of P, denoted by $H(P)$, is the union of all sides of P. We will also say that a tile t is on the i-th side of P if one of the sides of t is included in the i-th side of P.*

We can now introduce the notion of adjacency graph associated to a tiling, which will be our first step in the definition of graphs in one-to-one correspondence with $2D$-gon tilings.

Definition 2 *Let T be a tiling of a given $2D$-gon $P = (V, M)$, and let $n = |T|$ be the number of tiles of this tiling. Let $\pi : T \to \{1, ..., n\}$ be a labelling of the tiles of T. The adjacency graph of T is the undirected graph $A(T) = (V_T, E_T)$ where $V_T = \{\pi(t), t \in T\}$ and $\{\pi(t), \pi(t')\} \in E_T$ if and only if t and t' have one side in common in T. See Figure 4 for an example.*

The adjacency graph of a $2D$-gon tiling already encodes much information. However, the fact that two tilings have the same adjacency graph does not imply that they are equivalent: for example, one can verify that the tilings T_1 and T_3 in Figure 2 have the same adjacency graph. In order to obtain a one-to-one correspondence between a set of graphs and the set of tilings of a $2D$-gon, we introduce now the *de Bruijn graph.*

Definition 3 *Let $A = (V, E)$ be the adjacency graph of a tiling T of a $2D$-gon P. The* de Bruijn graph *$A' = (V, E, \lambda, \nu)$ is a graph with labelled vertices and with a distinguished vertex ν. The label $\lambda(t)$ of $t \in V$ is the pair of integers $\{i, j\}$ such that*

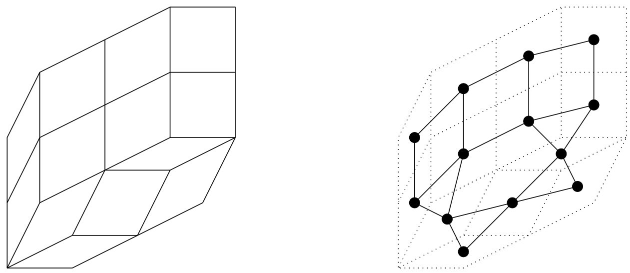

Figure 4: A tiling and its adjacency graph.

the two de Bruijn lines which contain the tile $\pi^{-1}(t)$ *are the* $i-th$ *and* $j-th$. *The vertex* ν, *called the* origin *of the graph, is associated to the tile with translation vector* $(0,...,0)$ *which is on the* $2D$*-th side of* P. *(See Figure 5 for an example).*

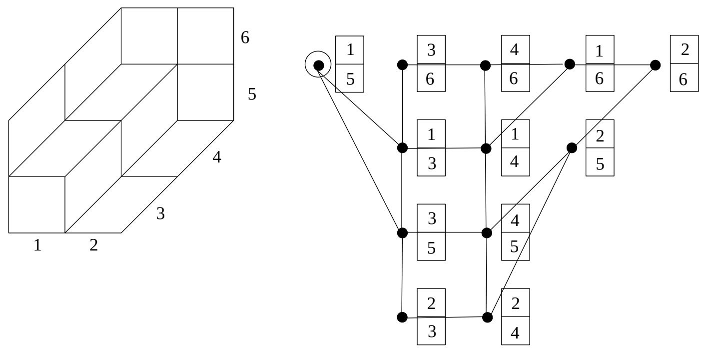

Figure 5: A tiling and its de Bruijn graph.

Theorem 1 *Given the de Bruijn graph of a tiling* T, *Algorithm 1 constructs a tiling equivalent to* T *in time* $O(n)$, *where* n *is the number of tiles of* T, *i.e. the number of vertices of the graph.*

This result shows that all the information contained in a $2D$-gon tiling is encoded in its de Bruijn graph. However, we will see that the de Bruijn graph contains much more information than really needed to construct the tiling. We will introduce the graph with origins of a tiling T, and we will see in the next section that this graph contains enough information to construct efficiently a tiling equivalent to T.

Definition 4 *Let* T *be a tiling of a* $2D$*-gon, and* $A=(V,E)$ *its adjacency graph. The* graph with origins *associated to* T *is* $G=(V,E,v_1,v_2)$, *where* v_1 *and* v_2 *are two vertices in* V *called the* origins *of* G *and defined as follows. Let* t_1 *be the tile of* T *on the* $2D$*-th side of* P *with translation vector* $(0,...,0)$. *Let* t_2 *be the tile on a side of* P *with translation vector* $(0,...,0,1)$ *if* t_1 *in also on the* $(2D-1)$*-th side of* P, *else the tile on a side of* P *with translation vector* $(0,...,0,1,1)$. *Then,* $v_1=\pi(t_1)$ *and* $v_2=\pi(t_2)$.

Algorithm 1: Construction of a tiling from its de Bruijn graph.

```
Input: G = (V, E, λ, ν), the de Bruijn graph of a tiling T.
Output: A tiling equivalent to T, given by a list of (tile, translation).
begin
    Let {i, j} = λ(ν) ;
    Set all the vertices as unmarked;
    resu ← {({f(i), f(j)}, (0, ..., 0))};
    current ← {(ν, (0, 0, ..., 0))};
    Mark ν ;
    while current ≠ ∅ ;
    do
        foreach τ = (v, trans) in current do
            foreach unmarked vertex v' such that (v, v') ∈ E ;
            do
                Let {i, j} be the label of v, and {j, k} be the label of v';
                Let trans' be a copy of trans;
                if not (f(i) > f(j) > f(k) or f(k) > f(j) > f(i)) then
                    Increase the f(i)-th component of trans' by one;
                resu ← resu ∪ {({f(j), f(k)}, trans')};
                current ← current ∪ {(v, trans')};
                Mark v';
            current ← current \ {v'};
    Return (resu);
end
```

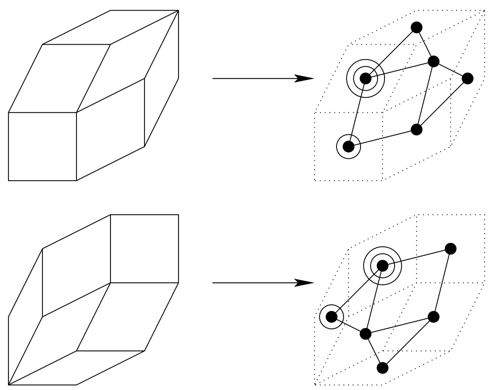

Figure 6: Two tilings and their graphs with origins. Notice that, if one removes the origins, then the two graphs are isomorphic.

Notice that the graphs with origins of the tilings T_1 and T_3 in Figure 2 are not isomorphic, as shown in Figure 6. We will show in the following that the correspondence between the graphs with origins we defined and the $2D$-gon tilings actually is one-to-one.

3 Duality

In this section, we give an algorithm which computes the de Bruijn graph of a tiling from its graph with origins. This correspondence is one-to-one, therefore, together with Algorithm 1 and Theorem 1 it shows that the graphs we introduced can be considered as *dual* of the considered tilings, despite the fact that they contain a very low amount of information: the graph with origins is reduced to the adjacency relation with two special marks. Our algorithm has complexity $O(n \cdot m)$, where n is the number of vertices of the graph, or equivalently the number of tiles of the tiling, and m is the sum of the multiplicities used to define the $2D$-gon.

In order to build the algorithm and prove its correctness, we will first prove some properties linking tilings of $2D$-gons and their graphs with origins. In particular, a special sub-structure, namely the *border*, will play a very important role. We introduce it now, first in the context of tilings, and then in the context of graphs.

Definition 5 *Let T be a tiling of a $2D$-gon P. Let $C \subseteq T$ be the set of tiles of T which have at least one point in $H(P)$. We define the* border *of T, denoted by $B(T) = (\pi(C), E)$ where $(\pi(t), \pi(t')) \in E$ if and only if t and t' have one side in common and if this common side has at least one point in $H(P)$. See Figure 7. Notice that this is a subgraph of the adjacency graph of T, but it is not the subgraph* induced *by C (some edges are missing).*

Definition 6 *Let $G = (V, E)$ be a graph. The border of G, denoted by $\mathcal{B}(G) = (\mathcal{V}, \mathcal{E})$, is the unique shortest self-avoiding cycle in G which contains all the vertices of G of degree at most 3, if it exists.*

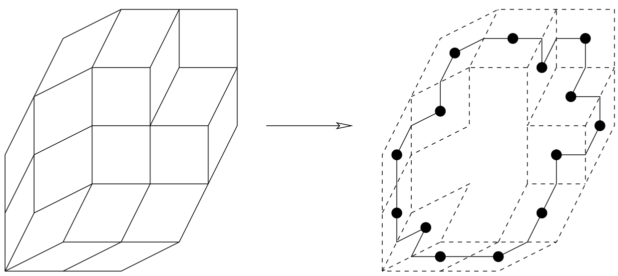

Figure 7: Left: a tiling. Right: its border and the border of the adjacency graph. As one may guess from this drawing, we will see below (Theorem 3) that the border of the tiling is nothing but the border of its adjacency graph.

The first step of algorithm to construct a tiling equivalent to a tiling T starting form the graph with origins of T will be to construct its border. We will first show that the border of T is nothing but the border of its adjacency graph, and then we will see how one can compute the border of this graph.

Definition 7 *Given a tiling T of a $2D$-gon P, a* fan *F of T is a k-tuple $(f_1, f_2, .., f_k)$ of tiles in T such that for all i between 1 and k:*

- *f_i and f_{i+1} have a side in common,*
- *$\cap_{i=1}^{k} f_i$ is a point of $H(P)$,*
- *f_1 and f_k have one side on the hull of the $2D$-gon.*

The tiles f_1 and f_k are called the extreme *tiles of F, and the integer $k-2$ is called the* size *of F. Moreover, we say that a fan F belongs to the $i-th$ side of P if one of its extreme tiles has a side included in the $i-th$ side of P.*

Lemma 1 *Let T be a tiling of a $2D$-gon, and t_1, t_2 be two tiles of T having exactly one point in common and having one side included in a given side of the $2D$-gon. The unique shortest path between $\pi(t_1)$ and $\pi(t_2)$ in $A(T)$ is $\pi(f_1) = \pi(t_1), \pi(f_2), ..., \pi(f_k) = \pi(t_2)$, where $\pi(f_i) \in B(T)$ for all i and $F = (f_1, ..., f_k)$ is a fan of T.*

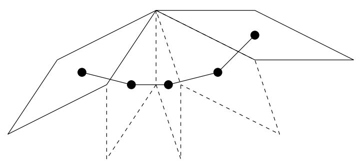

Figure 8: A fan and the unique shortest path between its extremities (Lemma 1).

Lemma 2 *Let $A = (V, E)$ be the adjacency graph of a tiling T of a $2D$-gon P. Let t be a tile of T on a side of P, and let $t' \neq t$ be a tile of T on a side of P such that $dist(\pi(t), \pi(t'))$ is minimal in A. Then, there is a fan $F = (f_1, ..., f_k)$ in T with $f_1 = t$ and $f_k = t'$.*

Theorem 2 *Let T be a tiling of a $2D$-gon $P = (V, M)$ with $M \neq \{1, 1, n\}$. The adjacency graph of T has a border.*

Theorem 3 *The border of a tiling and the border of its adjacency graph are isomorphic.*

We have now all the preliminary results necessary to write an algorithm which constructs the border of the adjacency graph of a $2D$-gon tiling (Algorithm 2), which, from Theorem 3, is equivalent to construct the border of the tiling.

Theorem 4 *Given the adjacency graph of a $2D$-gon tiling, Algorithm 2 computes its border in time $O(n)$ where n is the number of vertices of the graph.*

We will now show that, when one knows the border of the graph with origins of a $2D$-gon tiling T, then one can construct the de Bruijn lines of T by computing shortest paths in the graph.

Theorem 5 *Let A be the adjacency graph of a tiling of a $2D$-gon. Let a and b be two vertices associated to the extremities of a given de Bruijn line L. There exists a unique shortest path in A between a and b: $a = \pi(t_1), \pi(t_2), ..., \pi(t_k) = b$ with $L = \{t_1, t_2, ..., t_k\}$.*

Lemma 3 *Let T be a tiling of a $2D$-gon, α and β be two tiles of T, α being on the side i and β being on the side $m + i$ of the $2D$-gon (where m is the sum of the multiplicities). If α and β are the extremities of two different de Bruijn lines, then there are at least two shortest paths between them in $A(T)$.*

Algorithm 2: Construction of the border of the adjacency graph of a $2D$-gon tiling.

```
Input: G = (V, E, v1, v2) the adjacency graph of a 2D-gon tiling.
Output: The border of G, B(G), as an ordered list of vertices.
begin
    Let B be the set of vertices in V of degree< 4;
    Let v' be an element of B such that dist(v1, v') is minimal;
    𝓑 ← shortest path from v1 to v';
    repeat
        v ← last element of B;
        ν ← element just before v in B;
        Let B' be a copy of B;
        repeat
            Let v' be an element of B' such that dist(v, v') is minimal;
            Remove v' from B';
        until dist(v, v') ≤ dist(ν, v') or v' = v1;
        Add the shortest path from v to v' at the end of 𝓑;
    until v' = v1;
    if the second element of 𝓑 with degree < 4 is not v2 then
        Reverse 𝓑;
    Return(𝓑);
end
```

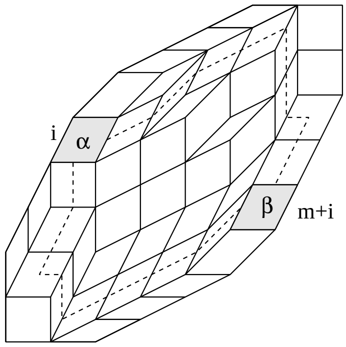

Figure 9: As announced in Lemma 3, there are two shortest paths between α and β.

Algorithm 3: Construction of the de Bruijn graph of a $2D$-gon tiling.

Input: $G = (V, E, v_1, v_2)$ the graph with origins of a $2D$-gon tiling T and $\mathcal{B}(G)$ its border.

Output: $G' = (V, E, \lambda, \nu)$, the de Bruijn graph of T.

```
begin
    i ← 1;
    foreach x ∈ B(G) do
        if d°(x) < 4 then
            B[i] ← x, i ← i + 1, N ← N + 1;
            if d°(x) = 2 then
                B[i] ← x, i ← i + 1, N ← N + 1;
    l ← 1;
    for α = 1 to N/2 do
        β ← α + N/2;
        build all the shortest paths between B[α] and B[β];
        if there is one path then
            foreach vertex v of the path do
                λ(v) ← λ(v) ∪ {l};
            l ← l + 1;
        else
            β ← β + 1;
    Return (V, E, λ, v1);
end
```

Theorem 6 *Given the graph with origins of a $2D$-gon tiling T and its border, Algorithm 3 computes the de Bruijn graph of T in time $O(n \cdot m)$ where n is the number of vertices of the graph and m is the sum of all the multiplicities which define the $2D$-gon.*

Starting from the graph with origins of a $2D$-gon tiling T, it is now clear that one can construct a tiling equivalent to T by computing the border of the graph with Algorithm 2, then compute the de Bruijn graph with Algorithm 3, and finally obtain the tiling using Algorithm 1. Therefore,we finally obtain the main result of this paper, by combining Theorems 1, 4 and 6:

Theorem 7 *Given the graph with origins of a $2D$-gon tiling T, there is an algorithm which constructs a $2D$-gon tiling equivalent to T in time $O(m \cdot n)$, where n is its number of tiles, and m the sum of the multiplicities of the vectors used to define the $2D$-gon.*

4 Perspectives

The algorithmic study of tilings of $2D$-gons is only at its beginning, and many open problems still exist. We cited the problem of knowing how many flips have to be done in order to obtain random tilings with a distribution *close* to the uniform distribution. Another important area is the generation of all the tilings of a $2D$-gon, and their enumeration. The encodings with graphs may be used to study these problems. For example, one may obtain a characterization of which graphs are the graphs associated to a $2D$-gon tiling : these graphs are planar, the degree of each vertex is at most four, and they may have many other properties which could help in generating and counting them.

Moreover, $2D$-gons are a special class (the dimension 2 case) of a very important class of objects, namely *zonotopes* [Zie95]. These objects can be viewed as generalizations of $2D$-gons in higher dimensions, and they play an important role in combinatorics and physics. They are also strongly related to matroid theory [RGZ94]. Many studies already deal with these objects, but their algorithmical manipulation is still a problem, while it would help a lot in verifying conjectures, compute special tilings, and compute some statistics over them. The results presented here may be extended to this more general case, leading to other classes of graphs with interesting properties. Notice however that this generalization is not obvious, since our proofs deeply use properties related to the dimension 2. It is well known in zonotopes theory that there is a gap of complexity between 2-dimensional zonotopes ($2D$-gons) and 3-dimensional ones [Zie95].

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A An application: random tilings

Tilings of $2D$-gons are an important model of quasicrystals in physics. In this context, it is very important to be able to sample random tilings, which helps the study of the entropy of the quasicrystal [BDMW02]. The sampling uses the key notion of *flip*: given a $2D$-gon tiling, one may rearrange locally three tiles (which form an hexagon) in order to obtain a new tiling of the same $2D$-gon (see Figure 10).This enables the generation of tilings of a $2D$-gon: it is shown in [Eln97, Ken93] that one can obtain all the tilings of a $2D$-gon from a given one by iterating the flip operation. When one wants to obtain a random tiling, one then has to choose a particular tiling and then iterate the flip operation *until the obtained tiling can be considered as random.* This notion of *when* one can stop the process is central when one wants to sample random tilings *with the uniform distribution.* It is possible to sample perfectly random tilings of hexagons because of the distributive lattice structure of the set of all the tilings [Pro98]. This technique can no longer be used for octagon, but a recent study explains how long the process has to be continued in order to be as close as one may want of the uniform distribution [Des01]. For the other $2D$-gons, *i.e.* when $D > 4$, there are no known results [BDMW02].

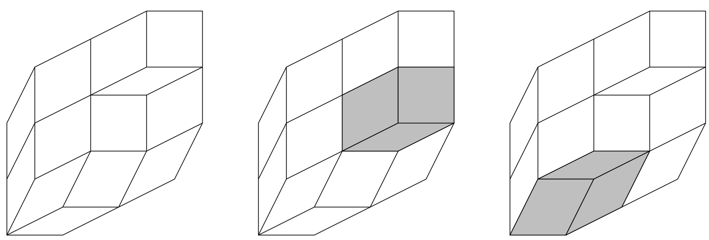

Figure 10: A tiling of an octagon ($D = 4$) (left) and two other tilings of the same octagon obtained from the first one by a flip (the shaded tiles are the ones which moved during the flip).

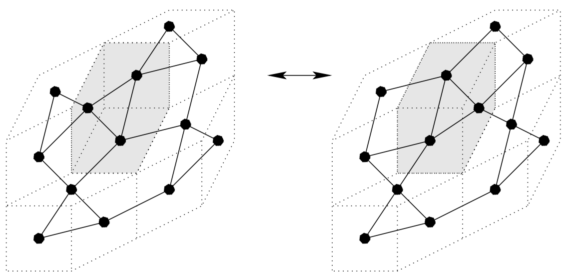

Figure 11: A flip on a $2D$-gon, and on its graph.

Therefore, when one wants to sample a random tiling of a given $2D$-gon P, the only solution is to construct a particular tiling of P and iterate the flip operation. To acheive this, one can use the graphs encodings we proposed above: the flip operation can be encoded on the graph, as shown in Figure 11. The vertices which correspond to the tiles to flip form a triangle in the graph, and conversly, all the triangles in the graph correspond to a possible flip in the tiling. Moreover, the transformation on the graph is a local rearrangement of vertices. Notice however that this transformation depends on the de Bruijn lines involved, and so it is simplier to use the de Bruijn graph than the graph with origins to compute it. We show in Figure 12 two random tilings of an octagon and a decagon obtained this way.

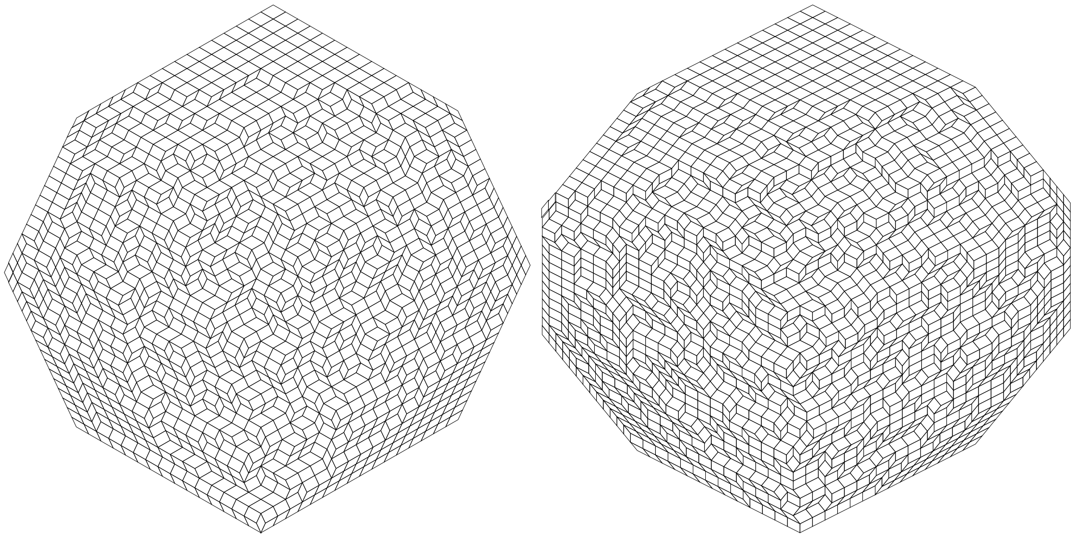

Figure 12: Left: a *random* tiling of a $16 \times 16 \times 16 \times 15$ octagon obtained after 1 million flips. Right: a *random* tiling of a $14 \times 14 \times 14 \times 14 \times 13$ decagon obtained after 1 million flips.